AFD Examples

Feiyang Chen

Lent 2021

Contents

Topic 1

Problem 1.1 Streamlines

Streamlines are curves which are instantaneously tangent to the velocity vector $\mathbf{u}$ of the flow, i.e. $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} \parallel \mathbf{u}$, or

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} \times \mathbf{u} = 0$$

for some parametrisation s of curves. If the flows are steady, we write the continuity equation as

$$\boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0$$

(a)

A flow is specified by

$$\mathbf{u} = a\hat{\phi} + b\hat{\mathbf{r}} = \begin{pmatrix} b \\ a \\ 0 \end{pmatrix}$$

in cylindrical coordinates, so the streamlines satisfy

$$\begin{aligned} \begin{pmatrix} -a\,\mathrm{d}z \\ b\,\mathrm{d}z \\ a\,\mathrm{d}r - b\,\mathrm{d}\phi \end{pmatrix} &= 0 \\ \implies \mathrm{d}z &= 0 \qquad \text{planar flow in } x\text{-}y \text{ plane} \\ \frac{\mathrm{d}r}{\mathrm{d}\phi} &= \frac{b}{a} \\ \boxed{r = \frac{b}{a}(\phi - \phi_0)} \end{aligned}$$

The density, if cylindrically symmetric, satisfies

$$\begin{aligned} \frac{1}{r}\frac{\partial (r\rho b)}{\partial r} &= 0 \\ \rho &\propto \frac{1}{r} \end{aligned}$$

(b)

Another flow is specified by

$$\mathbf{u} = aR^2\hat{\phi} + bR^2\hat{\mathbf{r}} = \begin{pmatrix} b \\ a \\ 0 \end{pmatrix}$$

so the streamlines satisfy

$$\begin{aligned} R^2 \begin{pmatrix} -a\,\mathrm{d}z \\ b\,\mathrm{d}z \\ a\,\mathrm{d}r - b\,\mathrm{d}\phi \end{pmatrix} &= 0 \\ \text{if } R \neq 0 \implies \mathrm{d}z &= 0 \qquad \text{planar flow in } x\text{-}y \text{ plane} \end{aligned}$$

$$\frac{\mathrm{d}r}{\mathrm{d}\phi} = \frac{b}{a} \qquad \text{as above}$$

$$\boxed{r = \frac{b}{a}(\phi - \phi_0)}$$

The density, if cylindrically symmetric, satisfies

$$\frac{1}{r}\frac{\partial(r\rho b r^2)}{\partial r} = 0$$

$$\rho \propto \frac{1}{r^3}$$

Both flows generate planar spiral streamlines. The difference is that the second flow allows an “axial” streamline along the z-axis.

Problem 1.2 Steady flow

Show that for a steady flow with $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$, the density ρ is constant along the streamlines. Need ρ be constant throughout the medium?

For a *steady* flow ($\frac{\partial \rho}{\partial t} = 0$), the continuity equation in Eulerian form reads

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0$$

$$\mathbf{u} \cdot \boldsymbol{\nabla}\rho + \rho \underbrace{\boldsymbol{\nabla} \cdot \mathbf{u}}_{0} = 0$$

$$\mathbf{u} \cdot \boldsymbol{\nabla}\rho = 0$$

The directional derivative of ρ along the tangent of any streamline, with $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} \parallel \mathbf{u}$, satisfies

$$\frac{\mathrm{d}\rho}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} \cdot \boldsymbol{\nabla}\rho \propto \mathbf{u} \cdot \boldsymbol{\nabla}\rho = 0$$

Therefore density is constant along all streamlines. However, it may still vary throughout the medium. A trivial example is a fluid which consists of multiple layers of static components of different densities which never interact with each other.

Problem 1.3 Transversal flow around a disc

Streamlines satisfy

$$\mathrm{d}\mathbf{r} \times \mathbf{u} = 0$$

Given

$$\mathbf{u} = U\left[\left(1+\frac{a^2}{R^2}\right)\hat{\mathbf{x}} - \frac{2a^2x}{R^3}\hat{\mathbf{R}}\right]$$

where $R^2 = x^2 + y^2$, so $\mathbf{u}$ is in the x-y plane. In this case, the streamlines all reside in planes as well. Using

$$\begin{aligned}\hat{R} &= \hat{x}\cos\phi + \hat{y}\sin\phi \\ \hat{\phi} &= -\hat{x}\sin\phi + \hat{y}\cos\phi\end{aligned}$$

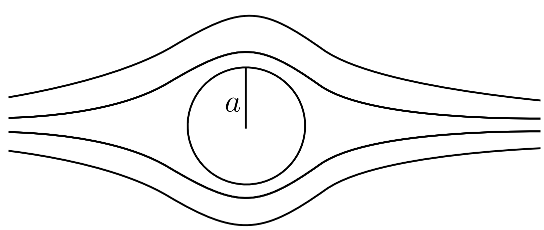
we realise

$$\hat{x} = \hat{R}\cos\phi - \hat{\phi}\sin\phi$$

and go back to the streamline equation

$$\begin{aligned}\begin{pmatrix}\mathrm{d}R \\ R\,\mathrm{d}\phi \\ 0\end{pmatrix} \times U\begin{pmatrix}\left(1+\frac{a^2}{R^2}\right)\cos\phi - \frac{2a^2\cos\phi}{R^2} \\ -\sin\phi\left(1+\frac{a^2}{R^2}\right) \\ 0\end{pmatrix} &= 0 \\ U\left(1-\frac{a^2}{R^2}\right)\cos\phi + U\left(\frac{\mathrm{d}R}{\mathrm{d}\phi} + \frac{a^2}{R^2}\frac{\mathrm{d}R}{\mathrm{d}\phi}\right)\sin\phi &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}\phi}\left[U\left(1-\frac{a^2}{R^2}\right)\sin\phi\right] &= 0\end{aligned}$$

The term in square brackets is therefore a constant along the streamline parametrised by ϕ.



rough plot of streamlines

This fluid velocity is a transverse laminar flow in $+x$ direction streaked by a disc (or cylinder) of radius a.

Problem 1.4 Radioactive dye

A steady 2D flow is described by $u_x = \frac{2}{x}$, $u_y = 1$. The streamlines satisfy

$$\begin{aligned} \frac{2}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - 1 &= 0 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x}{2} \\ y &= \frac{x^2}{4} + y_{\mathrm{init}} \end{aligned}$$

The steady surface density $\Sigma(x, y)$ satisfies continuity equation

$$\boldsymbol{\nabla}_{2D} \cdot (\Sigma \mathbf{u}) = 0$$

If Σ can be expressed as $\Sigma = \sigma_x(x)\sigma_y(y)$

$$\begin{aligned} \frac{1}{\sigma_x}\frac{\mathrm{d}}{\mathrm{d}x}(\sigma_x u_x) &= -\frac{1}{\sigma_y}\frac{\mathrm{d}}{\mathrm{d}y}\left(\sigma_y u_y\right) = -a = \text{const.} \\ \sigma_y &= \exp(ay) \\ \sigma_x &= \Sigma_0 \frac{x}{2}\exp\left(-\frac{ax^2}{4}\right) \\ \implies & \boxed{\Sigma = \frac{\Sigma_0}{2} x \exp\left[a\left(y - \frac{x^2}{4}\right)\right]} \end{aligned}$$

The density along a streamline can be alternatively expressed as

$$\Sigma = \frac{\Sigma_0}{2} x \exp(a y_{\mathrm{init}})$$

where y_{init} is the y-intercept when the streamline passes $x = 0$.

A radioactive inkblot is introduced in a small patch at (x_0, y_0). The nuclei decay such that their ***number per unit mass of sample*** *is $Q = Q_0 e^{-t}$.* Consequently the number per unit area, as the sample travels along the streakline (which coincides with the streamline for this steady flow) is

$$\begin{aligned} N &= Q_0 e^{-t}\frac{\Sigma_0}{2} x \exp\left[a(y_0 - \frac{x_0^2}{4})\right] \\ N &= \underbrace{Q_0 \frac{\Sigma_0}{2} \exp\left[a(y_0 - \frac{x_0^2}{4})\right]}_{\text{constant } N_0} e^{-t} x \end{aligned}$$

At a maximum number per area along the streakline,

$$\frac{1}{N_0}\frac{\mathrm{d}N}{\mathrm{d}t} = -e^{-t}x + e^{-t}u_x = 0$$
$$\left(\frac{x}{2} - x\right)e^{-t} = 0$$
$$x = \pm\sqrt{2}$$

Since $\mathrm{sgn}(u_x) = \mathrm{sgn}(x)$, the fluid elements always flow away from $x = 0$. In order that $|x| = \sqrt{2}$ is passed, $|x_0| < \sqrt{2}$. The maximum is located at

$$(x, y) = \left(\sqrt{2}, y_0 - x_0^2 + 2\right)$$

Problem 1.5 Curlless flow

(a)

$$\left[\mathbf{b} \times (\boldsymbol{\nabla} \times \mathbf{b})\right]_i = \epsilon_{ijk} b_j \epsilon_{kmn} \partial_m b_n$$
$$\left[\mathbf{b} \times (\boldsymbol{\nabla} \times \mathbf{b})\right]_i = \left(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right) b_j \partial_m b_n$$
$$\left[\mathbf{b} \times (\boldsymbol{\nabla} \times \mathbf{b})\right]_i = b_j \partial_i b_j - b_j \delta_j b_i$$
$$\mathbf{b} \times (\boldsymbol{\nabla} \times \mathbf{b}) = \frac{1}{2}\boldsymbol{\nabla}(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{b} \cdot \boldsymbol{\nabla})\mathbf{b}$$

(b)

$$\left[\boldsymbol{\nabla} \times (\boldsymbol{\nabla} a)\right]_i = \underbrace{\epsilon_{ijk}}_{\text{antisymmetric}} \underbrace{\partial_j \partial_k}_{\text{symmetric}} a$$
$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} a) = 0$$

(c)

$$\left[\boldsymbol{\nabla} \times (a\mathbf{b})\right]_i = \epsilon_{ijk} \partial_j (a b_k)$$
$$\left[\boldsymbol{\nabla} \times (a\mathbf{b})\right]_i = a\epsilon_{ijk}\partial_j b_k + \epsilon_{ijk}\left(\partial_j a\right) b_k$$
$$\boldsymbol{\nabla} \times (a\mathbf{b}) = a\boldsymbol{\nabla} \times b + (\boldsymbol{\nabla} a) \times \mathbf{b}$$

$\mathbf{g}$ is a conservative field. Using all of the above in the curl of the momentum equation, and assuming a barotropic equation of state $p = p(\rho) \implies \boldsymbol{\nabla} p = \frac{\mathrm{d}p}{\mathrm{d}\rho}\boldsymbol{\nabla}\rho$

$$\boldsymbol{\nabla} \times (-\boldsymbol{\nabla} p + \rho\mathbf{g}) = \boldsymbol{\nabla} \times \left(\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\right)$$

$$
\begin{aligned}
\rho\boldsymbol{\nabla}\times\left(\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\right)+(\boldsymbol{\nabla}\rho)\times\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}&=\boldsymbol{\nabla}\rho\times\mathbf{g}\\
\rho^2\boldsymbol{\nabla}\times\left(\frac{\partial\mathbf{u}}{\partial t}+\boldsymbol{\nabla}\left(\frac{1}{2}u^2\right)-\mathbf{u}\times\boldsymbol{\nabla}\times\mathbf{u}\right)+\rho(\boldsymbol{\nabla}\rho)\times\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}&=\boldsymbol{\nabla}\rho\times\left(\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}+\boldsymbol{\nabla}p(\rho)\right)\\
\rho^2\boldsymbol{\nabla}\times\left(\frac{\partial\mathbf{u}}{\partial t}-\mathbf{u}\times\boldsymbol{\nabla}\times\mathbf{u}\right)&=\boldsymbol{\nabla}\rho\times\frac{\mathrm{d}p}{\mathrm{d}\rho}\boldsymbol{\nabla}\rho\\
\frac{\partial}{\partial t}\boldsymbol{\nabla}\times\mathbf{u}&=\boldsymbol{\nabla}\times(\mathbf{u}\times\boldsymbol{\nabla}\times\mathbf{u})
\end{aligned}
$$

$\boldsymbol{\nabla}\times\mathbf{u}=0$ is a stationary solution of the differential equation for the field $\boldsymbol{\nabla}\times\mathbf{u}$. Thence the fluid velocity will remain curlless if it started in this state.

Problem 1.6 Self-gravitating slab

A static infinite slab of incompressible self-gravitating fluid of density ρ occupies the region $|z|<a$. Find the gravitational field everywhere and the pressure distribution within the slab.

The gravitational field has the same symmetries as ρ. Poisson's equation can be integrated to get

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\Psi)&=4\pi G\rho(\mathbf{r})\\
\oint\mathrm{d}\mathbf{S}\cdot(\boldsymbol{\nabla}\Psi)&=4\pi G\int_{\mathrm{encl}}\mathrm{d}V\,\rho(\mathbf{r})\\
2A|\boldsymbol{\nabla}\Psi|&=4\pi G\rho\begin{cases}2Az & |z|<a\\ 2Aa & |z|\geq a\end{cases}\\
-\boldsymbol{\nabla}\Psi&=-4\pi G\rho\begin{cases}\mathbf{z} & |z|<a\\ a\hat{\mathbf{z}} & |z|\geq a\end{cases}
\end{aligned}
$$

where we constructed an auxiliary surface with two faces of area A parallel to x-y plane, enclosing volume $2zA$.

The pressure distribution can be found via the equation of hydrostatic equilibrium

$$\frac{1}{\rho}\boldsymbol{\nabla}p=-\boldsymbol{\nabla}\Psi$$

Outside the slab where $\rho=0$, the pressure remains constantly 0. Within the slab,

$$
\begin{aligned}
\boldsymbol{\nabla}p&=-4\pi G\rho^2\mathbf{z}\\
p&=2\pi G\rho^2\left(a^2-z^2\right)
\end{aligned}
$$

If a galactic disk is approximated by a uniform density slab with density $1 \times 10^{-18}\ \mathrm{kg\,m^{-3}}$ *and* $a = 1 \times 10^{18}$ m*, determine the velocity of a star at the midplane if it starts from rest at* $z = a$*, and the period of its oscillation.*

The equation of motion of the star is

$$-4\pi G\rho\mathbf{z} = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}$$

with initial condition $z = a$, $v = 0$. Therefore it undergoes SHM with frequency

$$\omega = \sqrt{4\pi G\rho}$$

Velocity amplitude is simply

$$v_{\mathrm{mid}} = \max(v) = \omega a = \sqrt{4\pi G\rho}a$$
$$v_{\mathrm{mid}} = \max(v) = 2.9 \times 10^4\ \mathrm{m\,s^{-1}}$$

and period can be found by

$$T = \frac{2\pi}{\omega}$$
$$T = 2.17 \times 10^{14}\ \mathrm{s}$$

Problem 1.7 Temperature of stellar wind

For an ideal monatomic gas

$$E = \rho\left(\frac{1}{2}u^2 + \Psi + \frac{3p}{2\rho}\right)$$

and

$$p = \frac{k_B}{\mu m_p}\rho T = D\rho T$$

A steady flow sets $\frac{\partial}{\partial t} \to 0$, so the momentum and continuity equations read

$$\rho(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla}p + \rho\mathbf{g} \qquad\qquad \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0$$

For an adiabatic flow, $\dot{Q}_{\mathrm{cool}} = 0$, the energy equation reads

$$\boldsymbol{\nabla}\cdot\left[(E + p)\mathbf{u}\right] = 0$$

Finally, spherical symmetry sets $\mathbf{u}$, $\mathbf{g}$, and $\boldsymbol{\nabla}$ to purely radial.

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\rho u\right) = 0 \implies \rho = \frac{\rho_0 u_0 a^2}{ur^2}$$

$$\rho u\frac{\mathrm{d}u}{\mathrm{d}r}+\frac{\mathrm{d}D\rho T}{\mathrm{d}r}+\rho\frac{\mathrm{d}\Psi}{\mathrm{d}r}=0 \implies \rho\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{u^2}{2}+DT+\Psi\right]+DT\frac{\mathrm{d}\rho}{\mathrm{d}r}=0$$
$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[\rho ur^2\left(\frac{u^2}{2}+\Psi+\frac{5p}{2\rho}\right)\right]=0 \implies \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{u^2}{2}+\Psi+\frac{5DT}{2}\right]=0$$

Subtract third line $\times\rho$ from second line, a 4th-order polynomial of T can be found.

$$DT\frac{\mathrm{d}\rho}{\mathrm{d}r}=\rho\frac{3D}{2}\frac{\mathrm{d}T}{\mathrm{d}r}$$
$$\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}=\frac{3}{2T}\frac{\mathrm{d}T}{\mathrm{d}r}$$
$$\frac{\rho}{\rho_0}=\left(\frac{T}{T_0}\right)^{\frac{3}{2}}$$
$$\frac{u_0^2a^4T_0^3}{2r^4T^3}+\frac{5D}{2}T=F-\Psi$$

If the star's gravitational field dominates over the self-gravitation and pressure of the fluid, the $5DT/2$ pressure term can be neglected

$$F=\frac{u_0^2}{2}-\frac{GM}{a}$$
$$T=T_0\left\{\frac{u_0^2a^4}{2r^4}\left[GM\left(\frac{1}{r}-\frac{1}{a}\right)+\frac{u_0^2}{2}\right]^{-1}\right\}^{\frac{1}{3}}$$

If u_0 is just the gravitational escape velocity, the constant of motion F would be zero. In this regime T will decay like $\frac{1}{r}$, until eventually F is not negligible, i.e.

$$r\ll\frac{GM}{F}: \qquad T\sim r^{-1}$$
$$r\gg\frac{GM}{F}: \qquad T\sim r^{-4/3}$$

The general polynomial satisfied by T is

$$\frac{u_0^2a^4T_0^3}{2r^4T^3}+\frac{5D}{2}T=F-\Psi$$

In the short r regime, both terms on the left hand side decay like $\frac{1}{r}$. In the long r regime, the first term remains constant and the second term decays. Therefore, if pressure is negligible compared to kinetic energy of stellar wind near a, it will remain so throughout the wind.

Problem 1.8

A particle is released at rest at radius R_0 from the centre of a body mass M.

(a)

(i) The body is a point mass.

(ii) The body is a uniform sphere of radius R_0.

Either way the initial gravitational field, thence acceleration g, is determined by Poisson's equation

$$\begin{aligned}\nabla^2\Psi &= 4\pi G\rho\\ -gS &= 4\pi GM\\ g &= -\frac{GM}{R_0^2}\end{aligned}$$

where S_{R_0} is the area of the surface which encloses the spherical volume of radius R_0.

(b)

(i) The body is a point mass.

$$\begin{aligned}-gS_r &= 4\pi GM\\ g = v\frac{\mathrm{d}v}{\mathrm{d}r} &= -\frac{GM}{r^2}\\ \frac{1}{2}v^2 &= GM\left(\frac{1}{r}-\frac{1}{R_0}\right)\\ \frac{\mathrm{d}r}{\mathrm{d}t} &= -\sqrt{2GM}\sqrt{\frac{1}{r}-\frac{1}{R_0}}\\ \int_{R_0}^{0}\sqrt{\frac{rR_0}{R_0-r}}\,\mathrm{d}r &= -\sqrt{2GM}\int\mathrm{d}t\\ \Delta t &= \sqrt{\frac{R_0}{2GM}}\int_0^{R_0}\sqrt{\frac{r}{R_0-r}}\,\mathrm{d}r\end{aligned}$$

let $r = R_0\sin^2\theta$

$$\begin{aligned}\Delta t &= \sqrt{\frac{R_0^3}{2GM}}\int_0^{\frac{\pi}{2}}\frac{2\sin^2\theta\cos\theta}{\cos\theta}\,\mathrm{d}\theta\\ \Delta t &= \sqrt{\frac{R_0^3}{2GM}}\frac{\pi}{2}\end{aligned}$$

(ii) The body is a uniform sphere of radius R_0.

$$
\begin{aligned}
-gS_r &= 4\pi G \frac{M}{R_0^3} r^3 \\
g = \frac{\mathrm{d}^2 r}{\mathrm{d}t^2} &= -\frac{GM}{R_0^3} r \\
\Delta t &= \sqrt{\frac{R_0^3}{GM}} \arcsin\left(\frac{R_0 - r}{R_0}\right) \\
\Delta t &= \sqrt{\frac{R_0^3}{GM}} \frac{\pi}{2}
\end{aligned}
$$

A multiplicative factor of $\sqrt{2}$ longer than the point mass scenario.

For a cluster of fixed density with radius larger than he initial position of the star, the mass enclosed M is proportional to R_0^3. Therefore, the time it takes the star to reach center is independent of the starting position.

However I couldn't imagine a circumstance in which all the "background stars" are fixed instead of comoving.

Problem 1.9

(a)

Approximating the Earth's atmosphere as a perfect static isothermal gas in uniform gravitational field, the momentum equation reads

$$
\begin{aligned}
\boldsymbol{\nabla} p &= \rho \mathbf{g} \\
\frac{R^* T}{\mu} \boldsymbol{\nabla} \rho &= \rho \mathbf{g} = -\rho g \hat{\mathbf{z}} \\
\rho = \rho(z) &= \rho_0 \exp\left(-\frac{\mu g}{R^* T} z\right) \\
n = n(z) &= n_0 \exp\left(-\frac{\mu g}{R^* T} z\right)
\end{aligned}
$$

The characteristic length scale is

$$
l_c = \frac{R^* T}{\mu g}
$$

The fluid assumption breaks down where

$$
\frac{1}{n\sigma} \gg l_c
$$

$$z \gg l_c \ln(n_0 \sigma l_c)$$
$$z \gg z_b = 8 \times 10^5 \text{ m}$$

The radius of Earth is 6.4×10^6 m. At the height z_b, gravitational acceleration will have declined to

$$\left(\frac{6.4}{6.4 + 0.8}\right)^2 \approx 80\%$$

its value at surface, which is quite significant. Both uniform gravity and fluid assumptions break down.

(b)

Earth's speed in the Sun's frame is

$$v = \sqrt{\frac{GM}{r_\odot}}$$

Consider Earth's atmosphere as up to region with pressure higher than p_{surface}/e. If the Earth runs into a cloud of stationary hydrogen, for the ram pressure to be comparable to atmosphere pressure, the number density of that cloud needs to be

$$\rho v^2 \sim \frac{p}{e}$$
$$n \sim \frac{R^*}{k_B \mu v^2} \frac{p}{e}$$
$$n \sim 3 \times 10^{22} \text{ m}^{-3}$$

i.e. roughly 0.1% the number density of atmosphere.

Problem 1.10

Assuming the sun is a static fluid, momentum equation and Poisson's equation lead to

$$\boldsymbol{\nabla} p = \rho \mathbf{g}$$
$$-\boldsymbol{\nabla} \cdot \mathbf{g} = 4\pi G \rho$$

Assume all quantities vary over a radial scale length of order the radius of the Sun. Not sure what this means, I assumed that ρ simply steadily decreases radially to 0, i.e.

$$\rho = \rho_0 \left(1 - \frac{r}{R_\odot}\right)$$

Variations of other quantities follow.

$$M(r) = \rho_0 \int_0 \mathrm{d}r \, 4\pi r^2 \left(1 - \frac{r}{R_\odot}\right)$$

$$
\begin{aligned}
M(r) &= 4\pi\rho_0 r^3\left(\frac{1}{3} - \frac{r}{4R_\odot}\right) \\
M_\odot &= \frac{\rho_0}{4}\frac{4\pi R_\odot^3}{3} = \frac{\rho_0}{4}V_\odot \\
g(r) &= -\frac{GM(r)}{r^2} \\
g(r) &= -4\pi\rho_0 Gr\left(\frac{1}{3} - \frac{r}{4R_\odot}\right) \\
p_0 &= \int_{R_\odot}^{0} \rho g \,\mathrm{d}r \\
p_0 &= \frac{4\pi}{3}\rho_0^2 G\int_0^{R_\odot} \left(1 - \frac{r}{R_\odot}\right) r\left(1 - \frac{3r}{4R_\odot}\right)\mathrm{d}r \\
p_0 &= \frac{4\pi}{3}\rho_0^2 G\int_0^{R_\odot} r - \frac{7}{4}\frac{r^2}{R_\odot} + \frac{3}{4}\frac{r^3}{R_\odot^2}\,\mathrm{d}r \\
p_0 &= \frac{4\pi}{3}\rho_0^2 G\frac{5}{48}R_\odot^2 \\
p_0 &= 4.4 \times 10^{14}\ \text{pa}
\end{aligned}
$$

If the Sun is supported mainly by gas pressure of proton $p = \frac{R^*}{\mu}\rho_0 T$, temperature at core is

$$T \sim 1 \times 10^7\ \mathrm{K}$$

which may be an underestimation because the solar core is not pure hydrogen.

If the Sun is supported mainly by radiation pressure $p = \frac{1}{3}aT^4$, temperature at core is

$$T \sim 3.6 \times 10^7\ \mathrm{K}$$

Topic 2

Problem 2.1 Incompressible planet

The maximum pressure under which a planet's composition can remain incompressible is p_0. Fo such an incompressible planet of radius R, the Poisson equation is

$$\begin{aligned}
-\boldsymbol{\nabla}\cdot\mathbf{g} &= 4\pi G\rho \\
4\pi g r^2 &= 4\pi G\frac{4\pi}{3}\rho r^3 \\
g &= \frac{4\pi G}{3}\rho r
\end{aligned}$$

where spherical symmetry was exploited. The momentum equation relates the maximum total mass to the pressure at core

$$\begin{aligned}
\boldsymbol{\nabla} p &= \rho\mathbf{g} \\
\frac{\mathrm{d}p}{\mathrm{d}r} &= -\rho g \\
\int_R^0 \frac{\mathrm{d}p}{\mathrm{d}r}\,\mathrm{d}r &= \int_0^R \frac{4\pi G}{3}\rho^2 r \\
p_0 &= \frac{4\pi G}{3}\rho^2\frac{R^2}{2} \\
R &= \sqrt{\frac{3p_0}{2\pi\rho^2 G}} \\
M &= \frac{4\pi}{3}\rho R^3 \\
M &= \frac{4\pi}{3}\rho\sqrt{\left(\frac{3p_0}{2\pi\rho^2 G}\right)^3} \\
M &= \frac{2}{3}\frac{\rho}{\sqrt{\rho^6}}\sqrt{\frac{1}{2\pi}\left(\frac{3p_0}{G}\right)^3} \\
M &= \frac{2}{3\rho^2}\sqrt{\frac{1}{2\pi}\left(\frac{3p_0}{G}\right)^3}
\end{aligned}$$

Problem 2.2 Isothermal ring

An equilibrium ring of isothermal fluid orbits a star with mass M^* at radius R. In the plane of the ring, mechanical equilibrium results from a balance of centrifugal force and the gravitational force of the central object; normal to the ring (ie. vertically) equilibrium is *between the vertical component of the gravitational force of the central object and vertical pressure gradients in the ring gas.*

Consider the vertical direction z.

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\underbrace{\frac{GM^*}{R^2}}_{\text{gravitational field strength}} \underbrace{\frac{z}{R}}_{\text{vertical component}} \rho$$

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{GM^*}{R^3}\rho z$$

If the ring consists of ideal gas,

$$p = \frac{R^*T}{\mu}\rho$$

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -2\underbrace{\left(\frac{GM^*\mu}{2R^3R^*T}\right)}_{z_0^{-2}} z\rho$$

$$\rho \propto \exp\left(\frac{z^2}{z_0^2}\right)$$

The e-folding length is

$$z_0 = \sqrt{\frac{2R^3R^*T}{GM^*\mu}}$$

In the azimuthal plane the gravitational force from center balances with the centrifugal force, so

$$\frac{GM^*}{R^2} = \Omega^2 R$$

substituting into z_0,

$$z_0 = \sqrt{\frac{2R^*T}{\mu\Omega^2}} = \frac{1}{\Omega}\sqrt{\frac{2R^*T}{\mu}}$$

For $z_0 \ll R$ to be satisfied, we need

$$T \ll T_{\text{critical}} = \frac{\mu\Omega^2}{2R^*}R^2$$

If R is the astronomical unit, and the period of the ring is one year, this temperature takes the value

$$T_{\text{critical}} = 5.4 \times 10^4\ \text{K}$$

Problem 2.3

(a)

If

$$\Psi = -\frac{GM}{\left(r^2+b^2\right)^{1/2}}$$

$$\begin{aligned}
\nabla^2\Psi &= \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Psi}{\mathrm{d}r}\right) \\
&= \frac{GM}{2r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{2r}{(r^2+b^2)^{3/2}}\right) \\
&= \frac{GM}{r^2}\left[\frac{3r^2}{(r^2+b^2)^{3/2}}\right] - \frac{3r^3\times 2r}{2(r^2+b^2)^{5/2}} \\
&= GM\frac{3(r^2+b^2)-3r^2}{(r^2+b^2)^{5/2}} \\
4\pi G\rho &= \frac{GMb^2}{(r^2+b^2)^{5/2}} \\
\rho &= \frac{Mb^2}{4\pi(r^2+b^2)^{5/2}} \propto \left(\frac{1}{\sqrt{r^2+b^2}}\right)^5 \propto \Psi^5
\end{aligned}$$

(b)

$$\begin{aligned}
\boldsymbol{\nabla} p &= \rho g \\
&= -\rho\boldsymbol{\nabla}\Psi \\
&= -\rho\frac{GMr}{(r^2+b^2)^{3/2}} \\
&= -3(GMb)^2\frac{r}{(r^2+b^2)^4}
\end{aligned}$$

If the pressure vanishes at infinities

$$\begin{aligned}
p(r) &= -3(GMb)^2\int_\infty^r \frac{r'}{(r'^2+b^2)^4}\,\mathrm{d}r' \\
&= \frac{(GMb)^2}{2}\frac{1}{(r^2+b^2)^3} \\
&= \frac{(GMb)^2}{2}\left(\frac{4\pi\rho}{3Mb^2}\right)^{6/5} = K\rho^{6/5}
\end{aligned}$$

so we have the equation of state is polytropic with $n = \frac{1}{6/5-1} = 5$.

$$K = \frac{(GMb)^2}{2}\left(\frac{4\pi}{3Mb^2}\right)^{6/5}$$

(c)

If the matter is isentropic, such that $\gamma = 6/5$, and consists of ideal gas, the internal energy per unit mass is

$$\mathcal{E} = C_V T$$

together with the ideal gas equation

$$p = \frac{R^*}{\mu}\rho T \qquad \Longrightarrow \qquad \mathcal{E} = \frac{C_V \mu p}{R^* \rho}$$

Since $C_P = C_V + \frac{R^*}{\mu}$,

$$\begin{aligned}\mathcal{E} &= \frac{C_V}{C_P - C_V}\frac{p}{\rho} \\ &= \frac{1}{\gamma - 1}\frac{p}{\rho} = \frac{5p}{\rho}\end{aligned}$$

The total internal energy is then integrated over all mass differentials

$$\begin{aligned}U &= \int_0^\infty 4\pi r^2 \mathcal{E}\rho \, \mathrm{d}r \\ &= 5\int_0^\infty 4\pi r^2 p \, \mathrm{d}r \\ &= 10\pi (GMb)^2 \int_0^\infty \frac{r^2}{(r^2+b^2)^3}\, \mathrm{d}r \\ \text{let } r = b\tan\theta \implies \qquad U &= 10\pi \frac{(GM)^2}{b}\int_0^{\pi/2} \frac{\tan^2\theta \sec^2\theta}{(\tan^2\theta + 1)^3}\, \mathrm{d}\theta\end{aligned}$$

Use previous definition of $K = \frac{(GMb)^2}{2}\left(\frac{4\pi}{3Mb^2}\right)^{6/5}$

$$\begin{aligned}U &= \left[10\pi \times 2 \times (12\pi)^{6/5}\int_0^{\pi/2} \cos^2\theta \sin^2\theta \, \mathrm{d}\theta\right]\frac{K}{b^3}(Mb^2)^{6/5} \\ U &= \left[\frac{5}{2}(12\pi)^{6/5}\pi^2\right] K M^{6/5} b^{-3/5}\end{aligned}$$

Problem 2.4

In the lectures it was derived

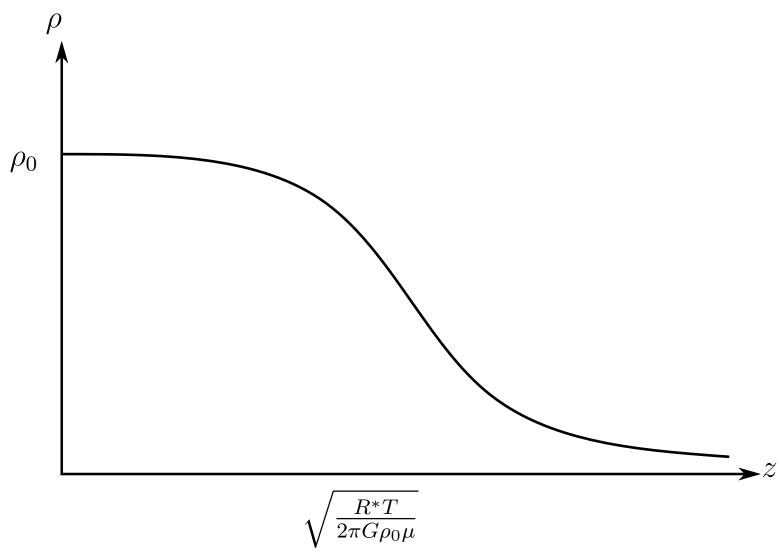
$$\rho = \frac{\rho_0}{\cosh^2\left(\sqrt{\frac{2\pi G \rho_0 \mu}{R^* T}}\, z\right)}$$

Near $z \to 0$, the density of the slab goes as

$$\rho \approx \rho_0 \left(1 - \frac{2\pi G \rho_0 \mu}{R^* T} z^2\right)$$

Near $z \to \infty$, it goes as

$$\rho \approx \rho_0 \exp\left(-2\sqrt{\frac{2\pi G \rho_0 \mu}{R^* T}} z\right)$$



A rough plot of ρ as a function of z. The relevant length scale is $\frac{1}{a} = \sqrt{\frac{R^* T}{2\pi G \rho_0 \mu}}$.

The gravitational field g satisfies

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{g} &= -4\pi G \rho \\
\frac{\mathrm{d}g}{\mathrm{d}z} &= -\frac{4\pi G \rho_0}{\cosh^2(az)} \\
g &= -\frac{G\rho_0}{a} \tanh(az)
\end{aligned}$$

where the boundary condition was determined by antisymmetry about $z = 0$. Thence we have

$$\begin{aligned}
\ddot{z} = g &= -\frac{4\pi G \rho_0}{a} \tanh(az) \\
\dot{z}\frac{\mathrm{d}\dot{z}}{\mathrm{d}z} &= -\frac{4\pi G \rho_0}{a} \tanh(az) \\
\dot{z}^2 &= -\frac{8\pi G \rho_0}{a^2} \ln\big(\cosh(az)\big) + C
\end{aligned}$$

Given a stars starts from rest at $z = z_0$,

$$\begin{aligned}\dot{z}^2 &= -\frac{8\pi G\rho_0}{a^2}\ln\bigl(\cosh(az)\bigr) + \frac{8\pi G\rho_0}{a^2}\ln\bigl(\cosh(az_0)\bigr)\\ &= \frac{8\pi G\rho_0 R^* T}{2\pi G\rho_0\mu}\ln\left(\frac{\cosh(az_0)}{\cosh(az)}\right)\\ &= \frac{4R^*T}{\mu}\ln\left(\frac{\cosh(az_0)}{\cosh(az)}\right)\end{aligned}$$

Problem 2.5

For a polytrope of index n,

$$\begin{aligned}&\int_0^\rho P(\rho')\rho'^{-2}\,\mathrm{d}\rho'\\ =&\int_0^\rho K\rho'^{1/n-1}\,\mathrm{d}\rho'\\ =&nK\rho^{1/n}\\ =&n\frac{P}{\rho}\end{aligned}$$

The internal energy per unit mass of an ideal gas, as shown in Problem 2.3.c is

$$\epsilon = \frac{1}{\gamma-1}\frac{P}{\rho}$$

Therefore,

$$\epsilon = \int_0^\rho P(\rho')\rho'^{-2}\,\mathrm{d}\rho' \iff \frac{1}{\gamma-1} = n \iff \boxed{\gamma = 1+\frac{1}{n}}.$$

Polytropes of index n satisfy Lane-Emden equation

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n$$

where $\rho = \rho_c\theta^n$, $\theta = \frac{\Psi_T-\Psi}{\Psi_T-\Psi_c}$, and $\xi = r\sqrt{4\pi G\rho_c/(\Psi_T-\Psi_c)}$. The equation of hydrostatic equilibrium gives

$$\rho = \left(\frac{\Psi_T-\Psi}{(n+1)K}\right)^n \implies \rho_c = \left(\frac{\Psi_T-\Psi_c}{(n+1)K}\right)^n \iff \Psi_t - \Psi_c = \rho_c^{1/n}(n+1)K$$

The Lane-Emden equation is not dependent on ξ, and the radius R of the star is defined at $\Psi = \Psi_T \implies \theta(\xi) = 0$, so the ξ-space interval that corresponds to the interior of the star is dependent only on the index n.

The total mass of a polytropic star is

$$\begin{aligned} M &= \int_0^R 4\pi r^2 \rho \,\mathrm{d}r \\ &= 4\pi\rho_c \left(\frac{\Psi_T - \Psi_c}{4\pi G \rho_c}\right)^{3/2} \int_0^{\xi_{\max}} \xi^2 \theta^n \,\mathrm{d}\theta \\ &= \rho_c^{\frac{1}{2}\left(\frac{3}{n}-1\right)} K^{3/2} \sqrt{\frac{(n+1)^3}{4\pi G^3}} \int_0^{\xi_{\max}} \xi^2 \theta^n \,\mathrm{d}\theta \\ &\propto \rho_c^{\frac{1}{2}\left(\frac{3}{n}-1\right)} \end{aligned}$$

and the total internal energy U is

$$\begin{aligned} U &= \int_0^R 4\pi r^2 \rho\epsilon \,\mathrm{d}r \\ &= 4\pi \left(\frac{\Psi_T - \Psi_c}{4\pi G \rho_c}\right)^{3/2} \int_0^{\xi_{\max}} \xi^2 (\rho_c \theta^n)^{1+1/n} \,\mathrm{d}\theta \\ &= 4\pi\rho_c^{1+1/n} \left(\frac{\Psi_T - \Psi_c}{4\pi G \rho_c}\right)^{3/2} \int_0^{\xi_{\max}} \xi^2 \theta^{n+1} \,\mathrm{d}\theta \\ &\propto \rho_c^{\frac{1}{2}\left(\frac{5}{n}-1\right)} \end{aligned}$$

The coefficients of proportionality are fully determined by K and n, so we can write

$$U = U_0 \left(\frac{M}{M_0}\right)^{\frac{5-n}{3-n}}$$

Problem 2.6

Using some results from Problem 2.5,

$$M = \rho_c^{\frac{1}{2}\left(\frac{3}{n}-1\right)} K^{3/2} \sqrt{\frac{(n+1)^3}{4\pi G^3}} \int_0^{\xi_{\max}} \xi^2 \theta^n \,\mathrm{d}\theta$$

$$\xi = r\sqrt{\frac{4\pi G}{n+1}\rho_c^{1-1/n} K^{-1}} \implies R = \xi_{\max}\sqrt{\frac{n+1}{4\pi G}} K^{1/2} \rho_c^{\frac{1}{2n}-\frac{1}{2}}$$

The temperature at core is simply

$$T_c = \frac{P_c \mu}{\rho_c R^*} = \frac{\mu}{R^*} K \rho_c^{1/n}$$

Ditch all the terms which are only functions of n or universal constants, let

$$M \propto R^l T_c^m$$

$$M \propto K^{l/2} \rho_c^{l/2n-l/2} K^m \rho_c^{m/n}$$
$$M \propto \rho_c^{\frac{1}{2}\left(\frac{3}{n}-1\right)} K^{3/2}$$
$$\implies \quad \frac{l}{2} + m = \frac{3}{2}; \qquad \frac{l}{2n} - \frac{l}{2} + \frac{m}{n} = \frac{3}{2n} - \frac{1}{2}$$
$$l = m = 1$$

So we get

$$M = \frac{M_0}{T_0 R_0} T_c R$$

For a series of stars which have the same polytropic index and central temperature, stellar mass is proportional to radius.

Problem 2.7

Take the continuity equation and make small perturbations about $p = p_0$, $\rho = \rho_0$, $\mathbf{u} = 0$,

$$\dot{\rho} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0$$
$$\frac{\mathrm{d}\Delta\rho}{\mathrm{d}t} + \rho_0 \boldsymbol{\nabla} \cdot \Delta\mathbf{u} = 0$$

In a plane wave solution, this reduces to

$$i\omega|\Delta\rho| - \rho_0 ik|\Delta u| = 0$$
$$|\Delta u| = \frac{|\Delta\rho|}{\rho_0}\frac{\omega}{k}$$

Recognizing the phase speed $\frac{\omega}{k}$, and $|\Delta\rho| \ll \rho_0$ for first order perturbations, there is

$$|\Delta u| \ll c_s$$

Speed of sound wave in air at s.t.p., which is fairly ideal

$$c_s = \sqrt{\frac{\mathrm{d}p}{\mathrm{d}\rho}} = \sqrt{\frac{R^* T}{\mu}} \approx 3 \times 10^2 \mathrm{~m\,s^{-1}}$$

The maximum longitudinal fluid velocity in the case of pressure fluctuations (which are proportional to density fluctuations in isothermal ideal gas) of 0.1% is

$$|\Delta u| \approx 0.3 \mathrm{~m\,s^{-1}}$$

Problem 2.8 Oblique shock

Let the direction normal to the shock front be x. Decompose the motion fluid velocity into parallel and perpendicular components to the normal of the shock front u and v. The continuity equation, integrated from just left to just right of the shock front, says

$$\rho_1 u_1 = \rho_2 u_2 \implies \frac{\rho_1}{\rho_2} = \frac{u_2}{u_1}$$

The momentum equation states

$$\partial_t(\rho u_i) = -\partial_j(\rho u_i u_j + p\delta_{ij}) + \rho g_i$$

The only direction along which ρ, p, or u changes discontinuously is x. In equilibrium, the left hand side is 0. Integrating from just left and just right to the shock front

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$
$$\rho_1 v_1 u_1 = \rho_2 v_2 u_2$$

Substituting in continuity equation, we find that there aren't any discontinuous changes of v. Finally conservation of energy gives

$$\frac{1}{2}u_1^2 + \mathcal{E}_1 + \frac{p_1}{\rho_1} = \frac{1}{2}u_2^2 + \mathcal{E}_2 + \frac{p_2}{\rho_2}$$

Given in this case the shock is adiabatic, internal energy is simply

$$\mathcal{E} = \frac{1}{\gamma - 1}\frac{p}{\rho}$$

so

$$\frac{1}{2}u_1^2 + \frac{\gamma p_1}{(\gamma-1)\rho_1} = \frac{1}{2}u_2^2 + \frac{\gamma p_2}{(\gamma-1)\rho_2}$$

The speed of sound in adiabatic medium is

$$c_s = \sqrt{\frac{\gamma p}{\rho}} \implies p = \frac{\rho c^2}{\gamma}$$

substituting this in all the ps in the equations above, get

$$\frac{1}{2}u_1^2 + \frac{c_1^2}{\gamma - 1} = \frac{1}{2}u_2^2 + \frac{c_2^2}{\gamma - 1}$$
$$\rho_1\left(u_1^2 + \frac{c_1^2}{\gamma}\right) = \rho_2\left(u_2^2 + \frac{c_1^2}{\gamma}\right) \implies u_2\left(u_1^2 + \frac{c_1^2}{\gamma}\right) = u_1\left(u_2^2 + \frac{c_2^2}{\gamma}\right)$$
$$\implies (\gamma - 1)\left[\frac{1}{2}(u_1^2 - u_2^2) + \frac{c_1^2}{\gamma - 1}\right] = c_2^2 = \gamma\left[u_1 u_2 + \frac{u_2 c_1^2}{u_1 \gamma} - u_2^2\right]$$

$$
\begin{aligned}
\frac{1}{2}\left(-u_1^2 + u_2^2\right) + c_1^2 \frac{u_1 - u_2}{u_1} &= -\frac{\gamma}{2}\left(u_1^2 - 2u_1u_2 + u_2^2\right) \\
-\frac{u_2 + u_1}{2} + \frac{c_1^2}{u_1} &= -\frac{\gamma}{2}(u_1 - u_2) \\
u_1 + u_2 - \frac{2c_1^2}{u_1} &= \gamma(u_1 - u_2) \\
u_2 &= \frac{1}{\gamma + 1}\left[\frac{2c_1^2}{u_1} + (\gamma - 1)u_1\right]
\end{aligned}
$$

An oblique adiabatic shock wave approaches the shock front at Mach number M, inclined to the normal at an angle θ. Its velocity component normal to the shock front is

$$u_1 = Mc_1 \cos\theta$$

After the shock, it leaves at angle $\theta + \chi$ to the normal. Since the components of the fluid velocity parallel to the plane are unchanged

$$
\begin{aligned}
u_1 \tan(\theta) &= u_2 \tan(\chi + \theta) \\
\cot(\chi + \theta) &= \cot(\theta)\frac{u_2}{u_1}
\end{aligned}
$$

$$\cot(\chi + \theta) = \frac{\cot(\theta)}{\gamma + 1}\left[\frac{2c_1^2}{u_1^2} + (\gamma - 1)\right] = \frac{1}{\gamma + 1}\frac{2 + (\gamma - 1)M^2\cos^2(\theta)}{M^2\cos(\theta)\sin(\theta)}$$

Use the trigonometric identity for $\cot(\chi + \theta)$.

$$
\begin{aligned}
\cot(\chi + \theta) &= \frac{\cot(\chi)\cot(\theta) - 1}{\cot(\chi) + \cot(\theta)} \\
\cot(\chi) &= \cot(\theta)\left[\frac{1}{\cos^2(\theta) - \cot(\chi + \theta)\cos(\theta)\sin(\theta)} - 1\right] \\
\cot(\chi) &= \cot(\theta)\left[\frac{(\gamma + 1)M^2}{(\gamma + 1)M^2\cos^2(\theta) - 2 - (\gamma - 1)M^2\cos^2(\theta)} - 1\right] \\
\cot(\chi) &= \cot(\theta)\left[\frac{(\gamma + 1)M^2}{2(M^2\cos^2(\theta) - 1)} - 1\right]
\end{aligned}
$$

Honestly though, what is the point of getting $\cot(\chi)$ except for psychological torture.

Problem 2.9

The lower bound of the interval that each cloud falls into the shock is if the two clouds do not decelerate at all upon collision.

$$t_{coll} \approx \frac{4R}{2v_0} = \frac{4 \times 3 \times 10^{16}}{4 \times 10^3} = 3 \times 10^{13}\ \mathrm{s}$$

The time scale of cooling is

$$t_{cool} \approx \frac{\frac{1}{2}u^2}{\dot{Q}} = 8 \times 10^{10}\ \mathrm{s}$$

This means that in the energy equation, the energy of the collided gas will be dissipated by the cooling process much sooner than the collision is complete. The shock is approximately isothermal.

The isothermal shock has $c_s^2 = u_1 u_2$. In the zero momentum frame, the clouds are each moving a speed $v_0 = 2 \times 10^3\ \mathrm{m\,s^{-1}}$. If after the shock the clouds are stationary in the zero momentum frame, we have in the shock front frame

$$u_2 = u_1 - v_0 = \frac{c_s^2}{u_1} = \frac{R^*T}{\mu u_1} \implies u_2 = 4.07 \times 10^1\ \mathrm{m\,s^{-1}} \qquad (\mu = 1)$$

u_2 is the speed of propagation of shock wave in ZMF. When the shock wave meets the outer edge of the clouds, the thickness of the shocked layer is

$$x = 2u_2 \frac{2R}{u_2 + v_0} = 2.39 \times 10^{15}\ \mathrm{m}$$

If the clouds later relaxes into a hydrostatic isothermal slab, quoting the results from problem 2.4,

$$\rho = \rho_0 \operatorname{sech}^2(ax)$$
$$m(x) = \frac{2\rho_0}{a} \tanh(ax)$$
$$m(x) = \sqrt{\frac{2\rho_0 R^* T}{\pi G \mu}} \tanh(ax) = \frac{aR^*T}{\pi G \mu} \tanh(ax)$$

where $m(x)$ is the column density of matter within distance x on both sides of the slab centre, thus the factor of a. Given $m(\infty) = 0.1\ \mathrm{kg\,m^{-2}}$,

$$a = m(\infty) \frac{\pi G \mu}{R^* T} = 2.53 \times 10^{-16}\ \mathrm{m^{-1}}$$
$$l_c = 3.96 \times 10^{15}\ \mathrm{m}$$

where l_c is the critical length scale of mass distribution of the slab. The accumulative mass fraction distribution is

$$\tanh(ax) = \tanh\left(\frac{x}{l_c}\right)$$

Topic 3

Problem 3.1

For an adiabatic fluid the energy and momentum equations are

$$\frac{1}{2}u_1^2 + \frac{\gamma p_1}{(\gamma-1)\rho_1} = \frac{1}{2}u_2^2 + \frac{\gamma p_2}{(\gamma-1)\rho_2} \qquad \frac{\rho_1}{\rho_2} = \frac{u_2}{u_1}$$

It was shown earlier that generally

$$u_2 = \frac{1}{\gamma+1}\left[\frac{2c_1^2}{u_1} + (\gamma-1)u_1\right]$$

In the limit of a strong shock $u_1 \gg c_1$, there is

$$u_2 = \frac{\gamma-1}{\gamma+1}u_1 \qquad \rho_1 = \frac{\gamma-1}{\gamma+1}\rho_2$$

and from the energy equation

$$\begin{aligned}
\frac{\rho_2 u_1^2}{p_2} + \frac{2\gamma}{\gamma-1}\frac{\rho_2}{\rho_1}\underbrace{\frac{p_1}{p_2}}_{\ll 1} &= \frac{\rho_2 u_2^2}{p_2} + \frac{2\gamma}{\gamma-1} \\
\frac{\rho_2}{p_2}\left(u_1^2 - u_2^2\right) = \frac{\rho_2}{p_2}\frac{\gamma^2+2\gamma+1-\gamma^2+2\gamma-1}{(\gamma-1)^2}u_2^2 &\approx \frac{2\gamma}{\gamma-1} \\
\frac{\rho_2 u_2^2}{p_2} &= \frac{\gamma-1}{2}
\end{aligned}$$

The Mach number is then

$$\begin{aligned}
M_2^2 &= \frac{u_2^2}{c_2^2} = \frac{\rho_2 u_2^2}{\gamma p_2} \qquad \Longleftarrow \; c_2 = \sqrt{\frac{\gamma p_2}{\rho_2}} \text{ for adiabatic fluid} \\
M_2^2 &= \frac{\gamma-1}{2\gamma}
\end{aligned}$$

For the sound speed ratio

$$\begin{aligned}
\frac{c_2}{c_1} &= \frac{u_2/M_2}{u_1/M_1} \\
&= \frac{u_2}{u_1}\frac{M_1}{M_2} \\
&= \frac{\gamma-1}{\gamma+1}\sqrt{\frac{2\gamma}{\gamma-1}}M_1
\end{aligned}$$

$$= \frac{\sqrt{2\gamma(\gamma-1)}}{\gamma+1}M_1$$

A shock from a supernova travelling through the surrounding interstellar medium is observed to be travelling with speed 3000 km/s. What is the temperature immediately behind the shock? A cavity is expanding so the interstellar medium is seen as incoming gas 1. The temperature in the expanding cavity T_2 is then

$$\frac{c_2}{c_1} = \sqrt{\frac{\gamma p_2}{\rho_2}\frac{\rho_1}{\gamma p_1}} = \sqrt{\frac{T_2}{T_1}}$$
$$\frac{T_2}{T_1} = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2}M_1^2$$
$$T_2 = T_1\frac{2\gamma(\gamma-1)}{(\gamma+1)^2}\frac{u_1^2\mu}{\gamma R^* T_1}$$
$$T_2 = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2}\frac{\mu}{\gamma R^*}u_1^2$$
$$T_2 = 1.73\times 10^8 \text{ m s}^{-1}$$

where we assumed $\mu = 1$, $R^* = 8.3\times 10^3$ J kg^{-1}, $\gamma = \frac{3}{2}$, and $u_1 = 3\times 10^6$ m s^{-1}.

Problem 3.2

The momentum equation under hydrostatic equilibrium and spherical symmetry

$$u\frac{\mathrm{d}u}{\mathrm{d}r} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r} - \frac{GM}{r^2} = -\frac{1}{\rho}\underbrace{\frac{\mathrm{d}p}{\mathrm{d}\rho}}_{c_s^2}\frac{\mathrm{d}\rho}{\mathrm{d}r} - \frac{GM}{r^2}$$

and conservation of mass outside the star (and no accretion of mass outside the star) means

$$\dot{M} = 4\pi r^2\rho u = \text{const. in space}$$

$$\frac{\mathrm{d}\ln\dot{M}}{\mathrm{d}r} = 2\frac{\mathrm{d}\ln r}{\mathrm{d}r} + \frac{\mathrm{d}\ln\rho}{\mathrm{d}r} + \frac{\mathrm{d}\ln u}{\mathrm{d}r} = 0$$
$$\frac{2}{r} + \frac{\mathrm{d}\ln\rho}{\mathrm{d}r} + \frac{\mathrm{d}\ln u}{\mathrm{d}r} = 0$$
$$\frac{2}{r} - \frac{1}{c_s^2}\left(\frac{GM}{r^2} + u^2\frac{\mathrm{d}\ln u}{\mathrm{d}r}\right) + \frac{\mathrm{d}\ln u}{\mathrm{d}r} = 0$$
$$\left(c_s^2 - u^2\right)\frac{\mathrm{d}\ln u}{\mathrm{d}r} = \frac{GM}{r^2} - \frac{2c_s^2}{r} = -\frac{2c_s^2}{r}\left(1 - \frac{GM}{2c_s^2 r}\right)$$

If the wind reaches isothermal sound speed $\sqrt{\frac{R^*T}{\mu}}$ the left hand side will vanish so

$$r_s = \frac{GM}{2c_s^2} = \frac{GM\mu}{2R^*T}$$

For $M = M_\odot$, $T = 2 \times 10^6$ K,

$$r_s = 4.02 \times 10^9 \text{ m}$$

which is about 13 lightseconds.

Problem 3.3

Quote the previous problem

$$\left(c_s^2 - u^2\right)\frac{\mathrm{d}\ln u}{\mathrm{d}r} = -\frac{2c_s^2}{r}\left(1 - \frac{GM}{2c_s^2 r}\right)$$

Use Bernoulli's principle

$$H = \frac{1}{2}u^2 + \int \frac{\mathrm{d}p}{\rho} - \frac{GM}{r} = \text{const.}$$

Assuming the gas remains isothermal, so $p = \frac{R^*}{\mu}T\rho$

$$\begin{aligned}
&H = \frac{1}{2}u^2 + \frac{R^*T}{\mu}\ln\rho - \frac{GM}{r} \\
\text{at } \infty \qquad &H = \frac{R^*T}{\mu} ln\rho_0 \\
&\rho_s = \rho_0 \exp\left[\frac{\mu}{R^*T}\left(\frac{GM}{r_s} - \frac{1}{2}c_s^2\right)\right] \\
&\rho_s = \rho_0 e^{3/2} \\
&r_s = \frac{GM}{2c_s^2} = \frac{GM\mu}{2R^*T} \\
&\dot{M} = 4\pi r_s^2 \rho_s c_s \\
&\dot{M} = \frac{4\pi G^2 M^2 \rho_0 e^{3/2}}{4c_s^4} c_s \\
&\dot{M} = \frac{\pi G^2 e^{3/2} \rho_0}{c_s^3} M^2
\end{aligned}$$

For $M = M_\odot$, $c_I = \sqrt{\frac{r^*T}{\mu}}$,

$$r_s = 9.53 \times 10^4 \text{ R}_\odot$$

$$\dot{M} = 8.38 \times 10^{14} \text{ kg s}^{-1}$$

To find the time elapsed before mass doubles solve

$$\dot{M} = \alpha M^2$$
$$\frac{1}{M_0} - \frac{1}{2M_0} = \alpha t_{\times 2}$$
$$t_{\times 2} = \frac{1}{2\alpha M_0} = \frac{M_\odot}{2\dot{M}_\odot} = 1.19 \times 10^{15} \text{ s} = 3.78 \times 10^7 \text{ years}$$

This time is inversely proportional to the initial mass of the star.

Problem 3.4

G, L, and ρ_0 cannot be combined to give a natural length or time scale of the problem, so the solution must be self-similar (hence the name similarity solutions?) on all length and time scales. Does that mean we must have a power law dependence for length evolution problems?

Let

$$r \propto L^a \rho_0^b t^c$$
$$\text{dimensional analysis} \implies \qquad a = \frac{1}{5}; \qquad b = -\frac{1}{5}; \qquad t = \frac{3}{5}$$

Assuming "the area occupied" is πr^2, the bubble is stalled when

$$\frac{\mathrm{d}r}{\mathrm{d}t} \propto L^a \rho_0^b \left(\frac{3}{5} t^{-2/5}\right) = c_s \implies t^{2/5} \propto L^a \rho_0^b c_s^{-1}$$

which means

$$\pi r^2 \propto L^{2a} \rho_0^{2b} \left(L^a \rho_0^b c_s^{-1}\right)^3 = L^{5a} \rho_0^{5b} c_s^{-3} = L \rho_0 c_s^{-3}$$

The area occupied by stalled bubbles is proportional to L, so if the total luminosity is fixed, the "porosity" of the galaxy is independent on how ionising stars are organised into clusters.

If the disc of a galaxy can be approximated by a uniform density gas slab with a sharp edge at height z, exceptionally luminous clusters of gases can escape the regime of similarity solution regime if their bubbles are not stalled until $r > z$.

Problem 3.5

Assume a barotropic equation of state, the convection instability condition is

$$\left(\frac{\partial \rho}{\partial z}\right)_K < \frac{\mathrm{d}\rho}{\mathrm{d}z} \qquad \text{(unstable)}$$

where fixing the K corresponds to adiabatically shifting a fluid element

$$\begin{aligned} p(z) &= K_1(z)\rho^\gamma \\ \left(\frac{\partial \rho}{\partial z}\right)_{K_1} &= \frac{1}{\gamma \rho^{\gamma-1} K_1}\frac{\mathrm{d}p}{\mathrm{d}z} = \frac{\rho}{\gamma}\frac{\mathrm{d}\ln p}{\mathrm{d}z} \\ \frac{\mathrm{d}\rho}{\mathrm{d}z} &= \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{p}{K_1}\right)^{\frac{1}{\gamma}} \\ \frac{\mathrm{d}\rho}{\mathrm{d}z} &= \frac{1}{\gamma}\left(\frac{p}{K_1}\right)^{\frac{1}{\gamma}-1}\left(\frac{1}{K_1}\frac{\mathrm{d}p}{\mathrm{d}z} - \frac{p}{K_1^2}\frac{\mathrm{d}K_1}{\mathrm{d}z}\right) \\ \frac{\mathrm{d}\rho}{\mathrm{d}z} &= \frac{\rho}{\gamma}\left(\frac{\mathrm{d}\ln p}{\mathrm{d}z} - \frac{\mathrm{d}\ln K_1}{\mathrm{d}z}\right) \\ \rho, \gamma > 0 \implies \qquad \frac{\mathrm{d}\ln K_1}{\mathrm{d}z} &< 0 \qquad \text{(unstable)} \end{aligned}$$

Given that the equation of state is polytropic

$$p = K_2\rho^{1+\frac{1}{n}} \implies K_1(z) = \frac{p}{\rho^\gamma} = K_2\rho^{1+\frac{1}{n}-\gamma} \implies \frac{\mathrm{d}\ln K_1}{\mathrm{d}z} = \left(1+\frac{1}{n}-\gamma\right)\frac{\mathrm{d}\ln p}{\mathrm{d}z}$$

$$\left(1+\frac{1}{n}-\gamma\right)\frac{\mathrm{d}\ln p}{\mathrm{d}z} > 0 \qquad \text{(stable)}$$

Since hydrostatic equilibrium requires $\frac{\mathrm{d}\ln p}{\mathrm{d}z} < 0$, the gas is stable iff

$$1 + \frac{1}{n} - \gamma < 0 \implies \frac{1}{n} < \gamma - 1$$

Using the condition derived in problem 3.5, we see the gas is convective stable for $\gamma > \frac{1}{2}$ which holds for all ideal gases.

Problem 3.6

The thermal instability condition is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}T}\left(\dot{Q}_{\text{heat}} - \dot{Q}_{\text{cool}}\right) &> 0 \qquad \text{(unstable)} \\ -\left(\frac{\partial \rho\sqrt{T}}{\partial T}\right)_p &> 0 \implies -\frac{\partial}{\partial T}\frac{p\mu}{R^*\sqrt{T}} > 0 \\ -\frac{1}{T^{3/2}} &> 0 \implies \text{condition always satisfied} \end{aligned}$$

At equilibrium,

$$\dot{Q}_{\text{heat}} = \dot{Q}_{\text{cool}} \implies \dot{Q}_{\text{Bremsstrahlung}} = \rho\sqrt{T}$$

$$p = \frac{R^* T}{\mu}\rho = \frac{R^*}{\mu}\dot{Q}^2_{\text{Bremsstrahlung}}\rho^{-1} \implies n = -\frac{1}{2}$$

Problem 3.7

In a uniform gaseous sphere containing one Jeans mass, the length scale of the sphere is

$$\lambda_J = \left(\frac{M_J}{\rho_0}\right)^{1/3}$$

Free fall is governed by a simple harmonic equation of motion

$$g = -\frac{4\pi G\rho_0 r}{3} \implies \omega_G = \sqrt{\frac{4\pi G\rho_0}{3}}$$

$$T_{\text{free fall}} = \frac{\pi}{\omega_G} = \sqrt{\frac{3\pi}{4G\rho_0}}$$

Sound wave crossing time for high wavenumber waves are

$$T_s = \frac{\lambda_J}{c_s}$$

$$T_s = \frac{\sqrt{\pi c_s^2}}{c_s\sqrt{G\rho_0}} = \sqrt{\frac{\pi}{G\rho_0}}$$

The ratio between these two time scales are

$$\sqrt{\frac{3}{4}} \sim 1$$

If such a sphere contracts homogeneously by $\mathfrak{R}$

$$R \to \mathfrak{R}^{-1}\sqrt{\frac{\pi c_s^2}{G\rho_0}}$$

Since the total mass is constant,

$$\rho = \frac{M}{R^3} \to \mathfrak{R}^3\rho_0 \qquad \lambda_J \to \mathfrak{R}^{-3/2}\sqrt{\frac{\pi c_s^2}{G\rho_0}}$$

such that

$$M_J = \rho\lambda_J^3 = \mathfrak{R}^{3-9/2}\rho_0\left(\frac{\pi c_s^2}{G\rho_0}\right)^{3/2} = \mathfrak{R}^{-3/2}M \implies M = \mathfrak{R}^{3/2}M_J$$

Problem 3.8

The momentum equation can be written as

$$\frac{\partial u}{\partial t} + \boldsymbol{\nabla}\left(\frac{1}{2}u^2\right) - u \times (\boldsymbol{\nabla} \times u) = -\boldsymbol{\nabla}\left(\int \frac{\mathrm{d}p}{\rho} + \Psi\right)$$

If the disc is rotating as a whole, none of the left-hand side has any z-components, so in the z-direction

$$\begin{aligned}
0 &= -\frac{\partial}{\partial z}\left(\int \frac{\mathrm{d}p}{\rho} + \Psi\right) \\
0 &= -\frac{\partial}{\partial z}\int \frac{\mathrm{d}p}{\rho} - \frac{GM}{r^2}\frac{z}{r} \\
0 &= -\frac{\partial}{\partial z}\int \frac{\mathrm{d}p}{\rho} - \frac{GM}{r^3}z
\end{aligned}$$

If $\rho = A(r)\left(z_m^2 - z^2\right)^2$

$$\begin{aligned}
\frac{GM}{r^3}z &= -\frac{\partial}{\partial z}\int \frac{\mathrm{d}p}{\rho} \\
\frac{\mathrm{d}z}{\mathrm{d}\rho}\frac{GM}{r^3}z &= -\frac{\mathrm{d}z}{\mathrm{d}\rho}\frac{\partial}{\partial z}\int \left(1 + \frac{1}{n}\right)K\rho^{\frac{1}{n}-1}\,\mathrm{d}\rho \\
-\frac{1}{4A(z_m^2 - z^2)z}\frac{GM}{r^3}z &= -\left(1 + \frac{1}{n}\right)K\rho^{\frac{1}{n}-1} \\
\frac{GM}{4r^3}\frac{1}{\sqrt{A}}\rho^{-\frac{1}{2}} &= \left(1 + \frac{1}{n}\right)K\rho^{\frac{1}{n}-1} \\
\implies \frac{1}{n} - 1 &= -\frac{1}{2} \implies n = 2 \\
\implies p &= K\rho^{\frac{1}{n}+1} \propto \left(z_m^2 - z^2\right)^3
\end{aligned}$$

The gas is stable iff

$$1 + \frac{1}{n} - \gamma < 0 \implies n > \frac{1}{\gamma - 1}$$

so the gas is stable against convection if composed of diatomic gas but is overstable if composed of monatomic gas.

Problem 3.9

Under hydrostatic equilibrium

$$\rho(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p$$

Across the jet-slab boundary the left hand side vanishes, so $p_s = p_j$.

$$p_j = \frac{R^*}{\mu} T_j \rho_j = p_s = \frac{R^*}{\mu} T_s \rho_s$$
$$\rho_j = \frac{T_s}{T_j} \rho_s = \frac{T_s}{T_j} \rho_0 \operatorname{sech}^2\left(\frac{z}{z_s}\right)$$

Inside the jet along z,

$$\rho_j u_z \frac{\mathrm{d}}{\mathrm{d}z} u_z = -\frac{\mathrm{d}}{\mathrm{d}z} p_j = -c_j^2 \frac{\mathrm{d}}{\mathrm{d}z} \rho_j$$
$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}z} u_z^2 = -c_j^2 \frac{\mathrm{d}}{\mathrm{d}z} \ln \rho_j$$
$$\frac{1}{2c_j^2} \left(u_z^2 - u_0^2\right) = \ln\left(\frac{\rho_{0j}}{\rho_j}\right)$$
$$u^2 = u_0^2 + 2c_j^2 \ln\left(\frac{\rho_{0j}}{\rho_j}\right)$$

Under isothermal conditions, Bernoulli's principle yields

$$u^2 = c_j^2 \left[1 + 2\ln\left(\frac{uA}{c_j A_m}\right)\right]$$
$$\frac{uA}{c_j A_m} = \exp\left[\frac{1}{2}\left(\frac{u^2}{c_j^2} - 1\right)\right] \qquad \Longrightarrow \; u_m = c_j$$
$$A_m = \frac{uA}{c_j} \exp\left[\frac{1}{2c_j^2}\left(c_j^2 - u^2\right)\right]$$
$$A_m = \frac{\dot{M}}{\rho_{0j} c_j} \exp\left[\frac{1}{2c_j^2}\left(c_j^2 - \left(\frac{\dot{M}}{A_0 \rho_{0j}}\right)^2\right)\right] \qquad \text{evaluated at reference point } z = 0$$

where $\dot{M} = \rho u A = \text{const.}$ and $c_j^2 = R^* T_j / \mu$.

Since $\dot{M} = \rho_s c_j A_m = \rho_{0j} u_0 A_0$

$$\rho_s = \frac{\dot{M}}{c_j A_m}$$
$$\rho_s = \rho_{0j} \exp\left[-\frac{1}{2c_j^2}\left(c_j^2 - \left(\frac{\dot{M}}{A_0 \rho_{0j}}\right)^2\right)\right] = \rho_{0j} \operatorname{sech}^2\left(\frac{z_m}{z_s}\right)$$

$$\implies z_m = z_s \operatorname{sech}^{-1} \exp\left[-\frac{1}{4c_j^2}\left(c_j^2 - \left(\frac{\dot{M}}{A_0\rho_{0j}}\right)^2\right)\right]$$

$$A(z) = \frac{\dot{M}}{\rho u} = \dot{M}\rho_{0j}^{-1} \cosh^2\left(\frac{z}{z_s}\right)\left(u_0^2 + 2c_j^2 \ln\left[\cosh^2\left(\frac{z}{z_s}\right)\right]\right)^{-1/2}$$

$$A(z) = A_0 \cosh^2\left(\frac{z}{z_s}\right)\left(1 + 2\frac{c_j^2}{u_0^2} \ln\left[\cosh^2\left(\frac{z}{z_s}\right)\right]\right)^{-1/2}$$

Topic 4

Problem 4.1

Under adiabatic jump conditions, immediately behind the shock

$$\frac{\rho u^2}{p} = \frac{1}{2}(\gamma - 1).$$

The momentum equation states

$$\underbrace{\rho u}_{\text{const.}} \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}p}{\mathrm{d}x} = 0 \implies \frac{\mathrm{d}}{\mathrm{d}x}\left(\rho u^2 + p\right) = 0$$

As ρ increases upon cooling down, ρu^2 decreases overall so p has to increase. Thence $\frac{\rho u^2}{p}$ decreases from $\frac{1}{2}(\gamma - 1)$.

For a monatomic gas, $\gamma = \frac{5}{3}$. Immediately after the shock

$$p_2 = \rho_2 u_2^2 \left[\frac{1}{2}(\gamma - 1)\right]^{-1} = 3\rho_2 u_2^2$$

Upon returning to pre-shock temperature, we could use the available relations for isothermal shock

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{u_1^2 \rho_1}{p_1} \implies p_2(T_1) = \rho_1 u_1^2 = 4\rho_2 u_2^2\Big|_{\text{immediately after shock}}$$

where we substituted

$$\frac{u_1}{u_2}\Big|_{\text{immediately after shock}} = \frac{\gamma + 1}{\gamma - 1} = 4$$

Therefore,

$$\frac{p_2(T_1)}{p_2^{\text{adiabatic}}} = \frac{4}{3}.$$

Assume constant thermal pressure from now on.

$$c_p \frac{\mathrm{d}T}{\mathrm{d}t} = -\mathcal{Q}^-$$

$$c_p \frac{\mathrm{d}T}{\mathrm{d}t} = -\underbrace{K(p)}_{\text{const.}} T^2$$

$$\boxed{T(t) = \frac{T_2}{1 + T_2 \frac{K}{c_p} t}}$$

$$p = R^* T \rho$$

$$\boxed{u = \frac{u_2 \rho_2}{\rho} = \frac{u_2 p R^* T}{R^* T_2 p} = \frac{u_2 T}{T_2}}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{u_2}{T_2} T$$

$$x = \int_0^t \frac{u_2}{1 + T_2 \frac{K}{c_p} t} \,\mathrm{d}t$$

$$x = \frac{c_p u_2}{K T_2} \ln\left(\frac{T_2}{T}\right)$$

$$\boxed{T = T_2 \exp\left(-\frac{K T_2}{c_p u_2}\right)}$$

Problem 4.2

Start from Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = -\frac{1}{\rho} \boldsymbol{\nabla} p + \nu \left[\nabla^2 \mathbf{u} + \frac{1}{3} \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u})\right]$$

Assume that $u_\phi = u_r = 0$ such that mass is not accreting onto annular surfaces of the pipe and the fluid is curlless. The fluid is incompressible, so under equilibrium we have $u_z = u_z(r)$.

$$0 = -\frac{1}{\rho} \boldsymbol{\nabla} p + \frac{\eta}{\rho} \nabla^2 \mathbf{u}$$

$$\frac{\mathrm{d}p}{\mathrm{d}z} = \eta \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r}\left(r \frac{\mathrm{d}u_z}{\mathrm{d}r}\right)$$

$$\frac{p_2 - p_1}{2\eta l r}\left(r^2 - A\right) = \frac{\mathrm{d}u_z}{\mathrm{d}r}$$

$$\frac{p_2 - p_1}{4\eta l}\left(r^2 - 2A \ln r - C\right) = u_z$$

$$\frac{p_2 - p_1}{4\eta l}\left(r^2 - \frac{R_1^2 - R_2^2}{\ln R_1 - \ln R_2} \ln \frac{r}{R_2} - R_2^2\right) = u_z$$

$$\int 2\pi r u_z \,\mathrm{d}r = \frac{p_2 - p_1}{4\eta l} \int_{R_1}^{R_2} 2\pi r \left(r^2 - \frac{R_1^2 - R_2^2}{\ln R_1 - \ln R_2} \ln \frac{r}{R_2} - R_2^2\right) \mathrm{d}r$$

$$\int 2\pi r u_z \,\mathrm{d}r = \frac{p_2 - p_1}{2\eta l} \pi \left[\frac{r^4}{4} - \frac{R_1^2 - R_2^2}{\ln R_1 - \ln R_2} R_2^2 \left(\frac{1}{2}\left(\frac{r}{R_2}\right)^2 \ln \frac{r}{R_2} - \frac{r^2}{4R_2^2}\right) - R_2^2 \frac{r^2}{2}\right]_{R_1}^{R_2}$$

$$\int 2\pi r u_z \,\mathrm{d}r = \frac{\pi}{2\eta l}(p_2 - p_1)\mathcal{I}$$

where

$$\mathcal{I} = \left[-\frac{R_2^4}{4} + \frac{R_1^2 - R_2^2}{\ln R_1 - \ln R_2}\frac{R_2^2}{4} - \frac{R_1^4}{4} + \left(R_1^2 - R_2^2\right)\frac{R_1^2}{2} - \frac{R_1^2 - R_2^2}{\ln R_1 - \ln R_2} R_1^2 \frac{1}{4} + \frac{R_2^2 R_1^2}{2} \right]$$

$$\mathcal{I} = \frac{1}{4}\left[R_1^4 - R_2^4 - \frac{(R_1^2 - R_2^2)^2}{\ln R_1 - \ln R_2} \right]$$

Finally, the total mass flux through the pipe is

$$\mathcal{Q} = \rho \int 2\pi r u_z \,\mathrm{d}r = \frac{\pi\rho}{8\eta l}(p_2 - p_1)\left[R_1^4 - R_2^4 - \frac{(R_1^2 - R_2^2)^2}{\ln\left(R_1/R_2\right)} \right]$$

Problem 4.3

Continuity equation requires $\mathbf{u}$ be divergence-free. For an incompressible fluid, Navier-Stoke's equation reads

$$\rho(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = \rho\mathbf{g} + \eta\nabla^2\mathbf{u}$$

where we have assumed barotropic equation of state such that the pressure gradient term vanishes with fixed ρ. There are not any fluid coming in or out the bounding planes so $u = u_x(z)$, where x and z are parallel and perpendicular to the planes

$$0 = \rho g \sin\alpha + \eta\partial_z^2 u_x$$

$$u_x = -\frac{g\sin\alpha}{\nu}\frac{z^2}{2} + Az + B$$

Applying B.C.s that $u_x = 0$ at $z = 0$ and $\frac{\mathrm{d}u_x}{\mathrm{d}z} = 0$ at $z = h$.

$$u_x = -\frac{g\sin\alpha}{\nu}\frac{z^2}{2} + \frac{gh\sin\alpha}{\nu}z$$

$$\mathcal{Q} = \rho\int_0^h u_x \,\mathrm{d}z$$

$$\mathcal{Q} = -\frac{\rho g\sin\alpha}{\nu}\frac{h^3}{6} + \frac{\rho g h\sin\alpha}{\nu}\frac{h^2}{2}$$

$$\mathcal{Q} = -\frac{\rho g h^3\sin\alpha}{3\nu}$$

Problem 4.4

IF there is no pressure gradient or gravitational field, for an initially unidirectional fluid $u_x(y)$

$$\frac{\partial\mathbf{u}}{\partial t} + \underbrace{\mathbf{u} \cdot \boldsymbol{\nabla}}_{0}\mathbf{u} = \nu\left[\nabla^2\mathbf{u} + \frac{1}{3}\boldsymbol{\nabla}\left(\underbrace{\boldsymbol{\nabla} \cdot \mathbf{u}}_{0} \right) \right]$$

$$\frac{\partial \mathbf{u}}{\partial t} = \nu \nabla^2 \mathbf{u} = \nu \nabla^2 u_x \hat{\mathbf{x}}$$

The direction of change is in the same direction x so the unidirectionality will be conserved.

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \nu \partial_y^2 u \\
\frac{\partial}{\partial t}\tilde{u} &= -\nu k^2 \tilde{u} \\
\tilde{u} &= \exp\bigl(-\nu k^2 t\bigr)\tilde{u}(k, t=0) = \exp\bigl(-\nu k^2 t\bigr)\int_{-\infty}^{\infty} u(y',0)\exp\bigl(-iky'\bigr)\,\mathrm{d}y' \\
u(y,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{u}\exp(iky)\,\mathrm{d}k \\
u(y,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} u(y',0)\int_{-\infty}^{\infty}\exp\left[-\nu k^2 t + ik(y'-y) + \frac{(y'-y)^2}{4\nu t}\right]\exp\left[-\frac{(y'-y)^2}{4\nu t}\right]\mathrm{d}k\,\mathrm{d}y' \\
u(y,t) &= \frac{1}{2\pi}\sqrt{\frac{\pi}{\nu t}}\int_{-\infty}^{\infty} u(y',0)\exp\left[-\frac{(y'-y)^2}{4\nu t}\right]\mathrm{d}y' \\
u(y,t) &= \frac{1}{2\sqrt{\pi\nu t}}\int_{-\infty}^{\infty} u(y',0)\exp\left[-\frac{(y'-y)^2}{4\nu t}\right]\mathrm{d}y'
\end{aligned}$$

Problem 4.5

In cylindrical symmetry, the continuity equation is

$$\begin{aligned}
\int \mathrm{d}z\,\frac{\partial \rho}{\partial t} + \int \mathrm{d}z\,\boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0 \\
\frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\Sigma u_R) = 0
\end{aligned}$$

and the Navier-Stokes equation is

$$\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla}p - \boldsymbol{\nabla}\Psi + \nu\left[\nabla^2\mathbf{u} + \frac{1}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u})\right] \\
\partial_t(R\Omega) + u_R\partial_R(R\Omega) + \frac{R\Omega u_R}{R} = \nu\frac{1}{R}\partial_R\bigl[R\partial_R(R\Omega)\bigr] - \nu\frac{R\Omega}{R^2} \\
\Sigma\partial_t\bigl(R^2\Omega\bigr) + \underbrace{R^2\Omega\partial_t\Sigma + \frac{R^2\Omega}{R}\frac{\partial}{\partial R}(R\Sigma u_R)}_{0} + \Sigma u_R R\bigl[\partial_R(R\Omega) + \Omega\bigr] = \Sigma\nu\Bigl\{\partial_R\bigl[R\partial_R(R\Omega)\bigr] - \Omega\Bigr\} \\
\partial_t\bigl(\Sigma R^2\Omega\bigr) + \partial_R\bigl(u_R\Sigma R^2\Omega\bigr) + \Omega\Sigma u_R R = \Sigma\nu\bigl(\partial_R R^2\partial_R\Omega + R\partial_R\Omega + \Omega - \Omega\bigr) \\
\partial_t\bigl(\Sigma R^2\Omega\bigr) + \frac{1}{R}\partial_R\bigl(u_R\Sigma R^3\Omega\bigr) = \frac{\Sigma\nu}{R}\partial_R\bigl(R^3\partial_R\Omega\bigr)
\end{aligned}$$

Problem 4.6

For non-relativistic scheme magnetic field is expected to dominate over electric field. To linear order of perturbation terms, plane wave solutions satisfy

$$\mathbf{j} = \frac{1}{\mu_0}\boldsymbol{\nabla} \times B \implies j = \frac{1}{\mu_0} ikB_1$$
$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{u}\right) = q\mathbf{E} + \mathbf{j} \times \mathbf{B} - \boldsymbol{\nabla} p \implies \rho_0(i\omega u_1) = \frac{1}{\mu_0}(ikB_1)B_0 + ikp_1$$
$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0 \implies i\omega\rho_1 - ik(\rho_0 u_1) = 0$$
$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{B}) \implies i\omega B_1 = iku_1 B_0$$

Substitute all the other three into the equation of motion for $\mathbf{u}$,

$$\omega u_1 = \left(\frac{B_0 B_1}{\rho_0 \mu_0} + \frac{p_1}{\rho_0}\right)k$$
$$\omega u_1 = \left(\frac{B_0 B_0 u_1 k}{\omega \rho_0 \mu_0} + \frac{p_1}{\rho_1}\frac{\rho_0 u_1 k}{\omega \rho_0}\right)k$$
$$\omega^2 = \left(\frac{B_0^2}{\rho_0 \mu_0} + \frac{\mathrm{d}p}{\mathrm{d}\rho}\right)k^2 = \left(v_A^2 + c_{s0}^2\right)k^2 \iff \frac{\partial^2 u_1}{\partial t^2} = \left(v_A^2 + c_{s0}^2\right)\frac{\partial^2 u_1}{\partial y^2}$$

The speed of plane wave found in magnetic material is greater than that in nonmagnetic material because there is an extra magnetic pressure contribution on top of the thermal pressure.

Problem 4.7

Jeans mass is tha mass when collapse time scale equals wave propagation time scale. It scales as

$$M_J \sim \rho_0 \lambda_J^3 = \rho_0 \left(\frac{\pi c_s^2}{G\rho_0}\right)^{3/2}$$

If supporting pressure is dominated by magnetism

$$c_s^2 \sim \frac{B^2}{\rho_0} \implies \lambda_J \sim \frac{B}{\rho_0}\frac{\sqrt{\pi}}{\mu_0\sqrt{G}} \implies M_J \sim \rho_0 \frac{B^3}{\rho_0^3} = \frac{B^3}{\rho_0^2}$$

If a uniform cloud contracts homogeneously like

$$r \to \alpha r$$

Conservation of mass leads to

$$\rho_0 r^3 \to \rho_0 r^3 \implies \rho_0 \to \alpha^{-3}\rho_0$$

The flux of a frozen-in magnetic field through the surface of the cloud is conserved throughout the motion, in this case contraction, of the cloud fluid

$$Br^2 \to Br^2 \implies B \to \alpha^{-2}B$$

Hence as the cloud collapses the number of magnetic Jeans masses contained

$$\frac{M}{M_J} \to \frac{M}{\frac{\alpha^{-6}}{\alpha^{-6}}M_J} = \frac{M}{M_J}$$

is conserved.