Electrodynamics and Optics Examples

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Contents

Topic 1

Problem 1.1

Shine unpolarised light (e.g. sunlight) onto a reflective surface at a nonzero angle of incidenece. The intensity of the reflected beam, viewed through the linear polariser, will be maximised when the transmitting axis is perpendicular to the plane of reflected beam.

Problem 1.2

$$\begin{aligned}
\mathbf{E}_{\pm} &= \mathbf{E}_1 \pm i\mathbf{E}_2 \\
\mathbf{E}_{\pm} &= e^{i(kz-\omega t)} \begin{pmatrix} E_1 \\ \pm iE_2 \\ 0 \end{pmatrix} \\
\mathbf{B}_{\pm} &= \frac{1}{\omega}\mathbf{k} \times \mathbf{E} \\
\mathbf{B}_{\pm} &= \frac{k}{\omega} e^{i(kz-\omega t)} \begin{pmatrix} \pm iE_2 \\ E_1 \\ 0 \end{pmatrix}
\end{aligned}$$

The instantaneous $\mathbf{E}_+$ field sweeps out a left-handed helix if $k_z > 0$.

k_z	$\mathbf{E}_+$	$\mathbf{E}_-$
> 0	LCP	RCP
< 0	RCP	LCP

Problem 1.3

$$J_\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \overbrace{\cos\theta}^{\text{Modulus of field strength projected along }\theta} \overbrace{\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}}^{\text{components in } x \text{ and } y}$$

$$J_\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sin\theta \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$J_\theta = J_\theta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}$$

Problem 1.4

When light is propagated through a uniaxial birefringent material, if the refractive index is different for different polarisation directions of the E field, the different polarisation components traverse different optical paths which are

$$\omega n_f \frac{d}{c} \qquad\qquad \omega n_s \frac{d}{c}$$

respectively, where d is the thickness along the direction of propagation. d can be varied so that the path difference is $\frac{\pi}{2}$, to make a quarter-wave plate. If the principal refractiveindices are n_o, n_o, n_e respectively, the minimum thickness required for a quarter-wave plate is[1]

$$\frac{\omega d|n_e - n_o|}{c} = \frac{\pi}{2} \implies d = \frac{\pi c}{2\omega|n_e - n_o|} = \frac{\lambda}{4|n_e - n_o|}$$

Problem 1.5

Jones vector of a general elliptically polarised 3:1 beam is

$$\mathbf{L}_i = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ \pm 3i \end{pmatrix}$$

Jones matrix of a quarter-wave plate is

$$J = \begin{pmatrix} 1 & \\ & i \end{pmatrix}$$

[1] This is too slim for calcite to support itself. In practice, two packs of birefringent material with their fast and slow axis aligned perpendicularly, and thickness difference equal to the calculated d_0 can be used to achieve a self-suppporting zero-order waveplate.

Output beam is thus

$$\begin{aligned}\mathbf{L}_o &= \begin{pmatrix}\cos\theta & -\sin\theta \\ i\sin\theta & i\cos\theta\end{pmatrix}\begin{pmatrix}1 \\ \pm 3i\end{pmatrix} \\ &= \begin{pmatrix}\cos\theta \mp 3i\sin\theta \\ i\sin\theta \mp 3\cos\theta\end{pmatrix}\end{aligned}$$

Any general linearly polarised beam has no phase difference in components of $\mathbf{L}$ so

$$\begin{gathered}(\cos\theta \mp 3i\sin\theta)(-i\sin\theta \mp 3\cos\theta) \in \mathbb{R} \\ \mp 3\cos^2\theta \mp 3\sin^2\theta - i\sin\theta\cos\theta + 9i\cos\theta\sin\theta \in \mathbb{R} \\ \cos\theta\sin\theta = 0 \implies \theta = \frac{n\pi}{2} \qquad \forall n \in \mathbb{Z}\end{gathered}$$

i.e. when the wave plate is alligned any integer multiple of $\frac{\pi}{4}$ to the incident beam a linearly polarised beam is produced. $\theta \to \theta + \pi$ is physically invariant, so $n = 0$ or 1 are the only meaningful values.

$$\mathbf{L}_o \propto \begin{pmatrix}1 \\ \mp 3\end{pmatrix} \; (n = 0) \qquad \text{or} \qquad \begin{pmatrix}\mp 3 \\ 1\end{pmatrix} \; (n = 1)$$

The possible polarisation directions of the output beam with respect to the major axis of the incident are

$$\pm\tan^{-1}\left(\frac{1}{3}\right) \; (n = 0) \qquad \text{or} \qquad \pm\tan^{-1}(3) - \frac{\pi}{2} = \mp\tan^{-1}\big(1/3\big) \; (n = 1)$$

The plus and minus signs depend on chirality of the incident beams.

Problem 1.6

The beam consists of polarised and unpolarised light. The intensity I_u of the unpolarised part is halved after passing through a linear polariser. The polarised part is elliptically polarised with major/minor axes vertical and horizontal, with Jones vector

$$\mathbf{L}_p = \begin{pmatrix}\pm i\sqrt{2 - \frac{I_u}{2}} \\ \sqrt{5 - \frac{I_u}{2}}\end{pmatrix}$$

When the beam is passed through a quarter-wave plate, the Jones vector of the polarised part is changed to

$$\mathbf{L}_o = \begin{pmatrix}i & \\ & 1\end{pmatrix}\begin{pmatrix}\pm i\sqrt{2 - \frac{I_u}{2}} \\ \sqrt{5 - \frac{I_u}{2}}\end{pmatrix} = \begin{pmatrix}\mp\sqrt{2 - \frac{I_u}{2}} \\ \sqrt{5 - \frac{I_u}{2}}\end{pmatrix}$$

Which is now linearly polarised. The unpolarised part is unaffected. Maximum intensity is found when the subsequent linear polariser is at angle 26.6° with the vertical, which means

$$\tan 26.6^\circ = \frac{\sqrt{2 - \frac{I_u}{2}}}{\sqrt{5 - \frac{I_u}{2}}}$$

$$\frac{I_u}{2} = 1 \text{ unit}$$

The intensity transmitted in this case is

$$\begin{aligned} I &= \underbrace{\left(1^2 + 2^2\right)}_{\text{Modulus of passed Jones vector squared}} + \underbrace{1}_{I_u/2} \\ &= 6.00 \text{ units} \end{aligned}$$

Before passing through the quarter-wave plate,

$$V_b = \frac{5}{5+2} = \frac{5}{7}$$

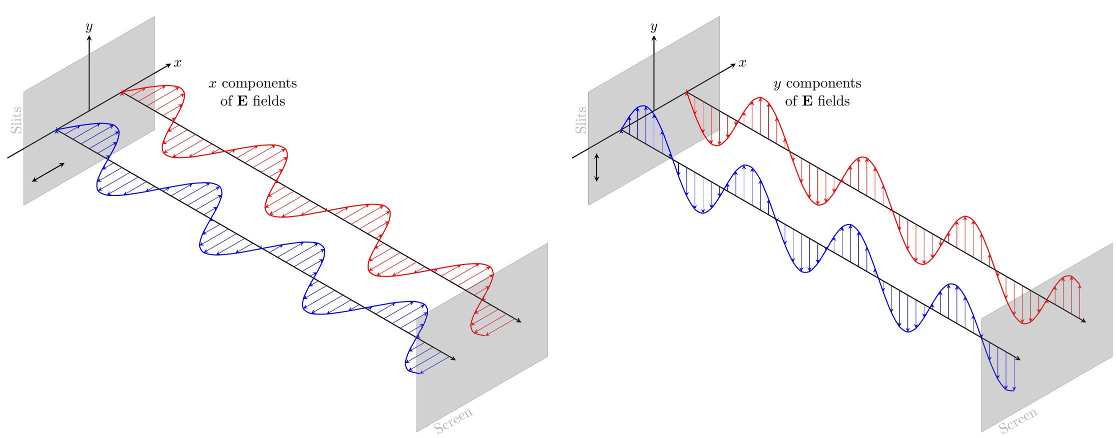
After passing through the plate,

$$V_a = \frac{5}{7} \qquad \text{still}$$

because the wave plate does not change the intensities of the polarised nor the unpolarised parts.

Problem 1.7

Circular polarisations of opposite handedness produce double slit interference fringes when observed through a plane polarising filter. When the plane polariser is rotated, we get the fringes gradually shifting in one direction. When the rotated angle reaches $\frac{\pi}{2}$, new maxima in the fringe have landed where minima used to be.



(a) Constructive interference in x (b) Destructive interference in y

Because of the opposite handedness, (say $(1,i)^T$ and $(1,-i)^T$) of light from the two slits, if at any point on screen the waves interfere constructively in x components ($L_x = 1+1$), they must interfere destructively in y components ($L_y = i - i$), vice versa.

Problem 1.8

(a)

The Jones vector of two ideal crossed linear polarisers can be written as

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

i.e. any light is eliminated.

(b)

The Jones matrix of a rotated waveplate is

$$\begin{aligned}
&\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & \\ & i \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\
=& \begin{pmatrix} \cos^2\theta + i\sin^2\theta & (1-i)\sin\theta\cos\theta \\ (1-i)\cos\theta\sin\theta & \sin^2\theta + i\cos^2\theta \end{pmatrix} \\
=& \begin{pmatrix} 1-(1-i)\sin^2\theta & (1-i)\sin\theta\cos\theta \\ (1-i)\cos\theta\sin\theta & (1-i)\sin^2\theta + i \end{pmatrix}
\end{aligned}$$

The resulting Jones matrix of this sandwiched waveplate is

$$\begin{aligned}
&\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1-(1-i)\sin^2\theta & (1-i)\sin\theta\cos\theta \\ (1-i)\cos\theta\sin\theta & (1-i)\sin^2\theta + i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
=& \begin{pmatrix} 0 & 0 \\ (1-i)\cos\theta\sin\theta & 0 \end{pmatrix} \\
\propto& \begin{pmatrix} 0 & 0 \\ \frac{1}{\sqrt{2}}\sin(2\theta) & 0 \end{pmatrix}
\end{aligned}$$

Which is consistent with our intuition: if $\theta = 0$ or $\pm\frac{\pi}{2}$ the resulting Jones matrix is trivial.

(c)

Unpolarised light has a uniform probabilty of pointing its electric field in any direction. When it is incident on a linear polariser, we have transmitted intensity

$$\begin{aligned} I_t &= \int_0^{2\pi} \frac{I}{2\pi} \left| \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \right|^2 \mathrm{d}\theta \\ &= \int_0^{2\pi} \frac{I}{2\pi} \cos^2\theta \, \mathrm{d}\theta \\ &= \frac{I}{2} \end{aligned}$$

Then the resulting transmission through the waveplate and the second polariser is not unlike that of polarised light

$$I_{\text{final}} = \frac{1}{2}\sin^2(2\theta)\frac{I}{2} = \frac{\sin^2(2\theta)}{4} I$$

which takes maximum value $\frac{I}{4}$ at $\theta = \frac{\pi}{4}$.

Problem 1.9

(a)

The nonmagnetic uniaxial crystal has

$$\mathbf{E} = \frac{1}{\varepsilon_0}\begin{pmatrix} \frac{1}{\varepsilon_o} & & \\ & \frac{1}{\varepsilon_o} & \\ & & \frac{1}{\varepsilon_e} \end{pmatrix}\mathbf{D} = \frac{D}{\varepsilon_0}\begin{pmatrix} \frac{-\cos\theta}{\varepsilon_o} \\ 0 \\ \frac{\sin\theta}{\varepsilon_e} \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \qquad \mathbf{B} = \mu_0\mathbf{H} = \mu_0 H e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\hat{\mathbf{j}}$$

These fields have

$$\begin{aligned} \nabla\cdot\mathbf{D} &= iD(-k_x\cos\theta + k_z\sin\theta)e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \\ \nabla\cdot\mathbf{B} &= \mu_0 k_y H e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \\ \nabla\times\mathbf{E} &= \frac{iD}{\varepsilon_0}\begin{pmatrix} \frac{k_y\sin\theta}{\varepsilon_e} \\ -k_z\frac{\cos\theta}{\varepsilon_o} - k_x\frac{\sin\theta}{\varepsilon_e} \\ k_y\frac{\cos\theta}{\varepsilon_o} \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \\ \nabla\times\mathbf{H} &= iH\begin{pmatrix} -k_z \\ 0 \\ k_x \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \end{aligned}$$

which satisfy Maxwell's equations provided that

$$\mathbf{k} = -\omega \frac{D}{H} \begin{pmatrix} \sin\theta \\ 0 \\ \cos\theta \end{pmatrix}$$

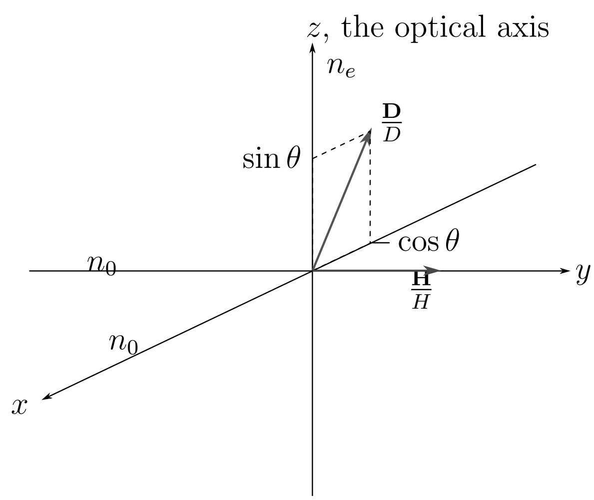
$$\frac{\omega D^2}{H\varepsilon_0}\left(\frac{\cos^2\theta}{\varepsilon_o} + \frac{\sin^2\theta}{\varepsilon_e}\right) = \omega\mu_0 H$$

$$\implies \frac{D^2}{H^2}\left(\frac{\cos^2\theta}{\varepsilon_o} + \frac{\sin^2\theta}{\varepsilon_e}\right) = c^{-2}$$

Rearranging, we get

$$\mathbf{k} = \pm\omega \frac{n_b}{c}\omega \begin{pmatrix} \sin\theta \\ 0 \\ \cos\theta \end{pmatrix}$$

where ${n_b}^{-2} \equiv \left(\frac{\cos^2\theta}{\varepsilon_o} + \frac{\sin^2\theta}{\varepsilon_e}\right)$



(b)

The Poynting vector is given by

$$\begin{aligned}\mathbf{N} &= \mathbf{E} \times \mathbf{H} \\ &= -\frac{DH}{2\varepsilon_0}\begin{pmatrix} \frac{\sin\theta}{\varepsilon_e} \\ 0 \\ \frac{\cos\theta}{\varepsilon_o} \end{pmatrix}\end{aligned}$$

which is at angle $\theta' = \arctan\left(\frac{\varepsilon_o}{\varepsilon_e}\tan\theta\right)$ to the optical axis.

Problem 1.10

If a molecules is composed of 4 *identical* polarisable spheres at corners of a regular tetrahedron, its mirror image cannot be distinguished from itself, so chirality is not present.

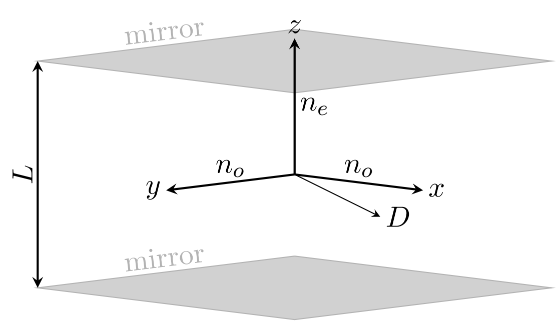
A triatomic molecule, consisting of spheres interacting by Coulomb fields, is always planar. A planar molecule does not demonstrate chirality either.

Problem 1.11

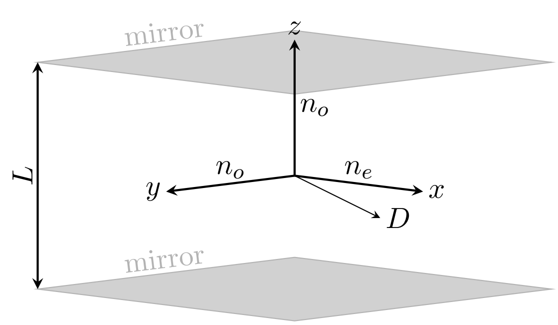
(a)

- With the optic axis aligned with the z axis, **D** field is in the degenerate plane, so the **E** vector will be parallel to **D** and hence also in the x-y plane, which gives us

$$\begin{aligned}\omega &= \frac{c}{n_o}k \\ &= \frac{\pi c}{L\, n_o}m \qquad m \in \mathbb{Z}\end{aligned}$$



- In this case both **D** and **E** are still in x-y plane, but the x and y components of the wave propagate at different speeds, for a stationary solution, we must have



$$\omega = \frac{\pi c}{L\, n_o} m = \frac{\pi c}{L\, n_e} n \qquad m, n \in \mathbb{Z}$$

simultaneously, unless $\mathbf{D}$ is parallel to one of the principal axes.

(b)

A chiral material is fully isotropic, and its eigenpolarisations (polarisations that remain coherent through propagation) are LCP and RCP.

Handedness of circularly polarised lights revert upon reflection. Therefore, the boundary condition is

$$\frac{\omega}{c/n_l} L + \frac{\omega}{c/n_r} L = \frac{2 n_o \omega L}{c} = 2m\pi$$
$$\omega = \frac{\pi c}{n_o \omega L}$$

(c)

The total angle that a plane polarised beam is (clockwise) rotated through by a return trip in this system is $2VB_0L$, and the phase change is $2kL + \pi$. The changes in polarisation direction and phase are fully described by the following complex matrix

$$-\exp(i2kL) \begin{pmatrix} \cos(2VB_0L) & \sin(2VB_0L) \\ -\sin(2VB_0L) & \cos(2VB_0L) \end{pmatrix}$$

The vanishing boundary condition at the mirror requires the above to have eigenvalue -1. i.e. λ has a solution -1 in the equation below

$$\left(\cos(2VB_0L) + e^{-i2kL}\lambda\right)\left(\cos(2VB_0L) + e^{-i2kL}\lambda\right) + \sin^2(2VB_0L) = 0$$
$$1 + 2e^{i2kL}\lambda \cos(2VB_0L) + e^{i4kL}\lambda^2 = 0$$
$$2\cos(2kL) = 2\cos(2VB_0L)$$
$$kL = \pm VB_0L + m\pi$$
$$\omega = \frac{c}{n}\left(\pm VB_0 + \frac{m\pi}{L}\right) \qquad m \in \mathbb{Z}$$

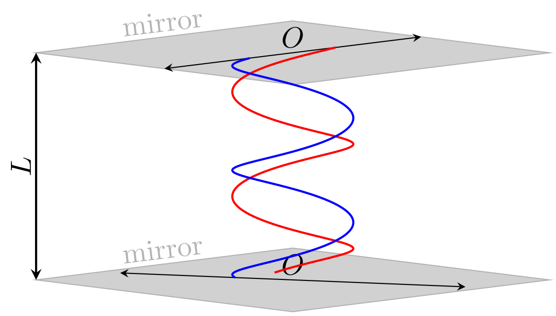


Figure 2: The boundary condition is that at both mirrors, the two waves superimpose to 0 at all times.

Problem 1.12

The equation of motion of each electron in the plasma is

$$m\ddot{\mathbf{r}} = -e(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B})$$

where the contribution from the magnetic field in the EM wave has been neglected. The transverse response of the electrons is thus given by

$$-\omega^2 \mathbf{r} = -\frac{e}{m}(\mathbf{E} - i\omega\mathbf{r} \times \mathbf{B})$$

LCP and RCP modes have respectively

$$\begin{aligned} \mathbf{r}_L &= a\begin{pmatrix}1\\ i\end{pmatrix} & \mathbf{r}_R &= a\begin{pmatrix}1\\ -i\end{pmatrix} \\ -i\omega\mathbf{r}_L \times \mathbf{B} &= \omega B_z \mathbf{r}_L & -i\omega\mathbf{r}_R \times \mathbf{B} &= -\omega B_z \mathbf{r}_R \\ \omega(\omega - \omega_c)\mathbf{r}_L &= \frac{e}{m}\mathbf{E} & \omega(\omega + \omega_c)\mathbf{r}_R &= \frac{e}{m}\mathbf{E} \\ \frac{1}{\varepsilon_0}\mathbf{P}_L &= -\frac{ne^2/m\varepsilon_0}{\omega(\omega-\omega_c)}\mathbf{E} & \frac{1}{\varepsilon_0}\mathbf{P}_R &= -\frac{ne^2/m\varepsilon_0}{\omega(\omega+\omega_c)}\mathbf{E} \\ \frac{1}{\varepsilon_0}\mathbf{P}_L &= -\frac{\omega_p^2}{\omega(\omega-\omega_c)}\mathbf{E} & \frac{1}{\varepsilon_0}\mathbf{P}_R &= -\frac{\omega_p^2}{\omega(\omega+\omega_c)}\mathbf{E} \\ \mathbf{D}_L &= \varepsilon_0\left[1 - \frac{\omega_p^2}{\omega(\omega-\omega_c)}\right]\mathbf{E} & \mathbf{D}_R &= \varepsilon_0\left[1 - \frac{\omega_p^2}{\omega(\omega+\omega_c)}\right]\mathbf{E} \end{aligned}$$

At low frequencies, $\omega \ll \omega_c$, $\omega \ll \frac{\omega_p^2}{\omega_c}$

$$\mathbf{D}_L \approx \varepsilon_0\left[1 + \frac{\omega_p^2}{\omega\omega_c}\right]\mathbf{E} \qquad \mathbf{D}_R \approx \varepsilon_0\left[1 - \frac{\omega_p^2}{\omega\omega_c}\right]\mathbf{E}$$

which gives us that the effective susceptibility of LCP will be (+)ve and that of RCP will be (−)ve, so the wavevector of LCP will be real, but that of RCP will be imaginary, i.e. RCP cannot propagate.

$$\begin{aligned} k &= \frac{n\omega}{c} = \frac{\sqrt{\varepsilon}}{c}\omega \\ &= \frac{\omega}{c}\frac{\omega_p}{\sqrt{\omega\omega_c}} \\ &= \frac{\omega_p}{c}\sqrt{\frac{\omega}{\omega_c}} \\ v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} &= \frac{2\sqrt{\omega_c\omega}}{\omega_p}c \end{aligned}$$

Problem 1.13

The dielectric multilayer has dispersion relation

$$\cos(qd) = F(\omega) = \cos(k_a a)\cos(k_b b) - \frac{1}{2}\left(\frac{k_b}{k_a} + \frac{k_a}{k_b}\right)\sin(k_a a)\sin(k_b b)$$

where $k_a = \frac{\omega n_a}{c}$ etc.

At low frequencies, we can calculate the effective refractive index $n_{\text{eff}} = \frac{qc}{\omega}$

$$\begin{aligned} \cos(qd) &= 1 + \frac{1}{2}k_a^2a^2 + \frac{1}{2}k_b^2b^2 - \frac{1}{2}\left(k_a^2ab + k_b^2ab\right) + O(\omega^4) \\ 1 - \frac{1}{2}(qd)^2 &= 1 + \frac{1}{2}k_a^2a^2 + \frac{1}{2}k_b^2b^2 - \frac{1}{2}\left(k_a^2ab + k_b^2ab\right) + O(\omega^4 + q^4) \\ (n_{\text{eff}}d)^2 &= n_a^2a(a+b) + n_b^2b(a+b) + O(\omega^2 + \frac{q^4}{\omega^2}) \\ n_{\text{eff}} &\approx \sqrt{\frac{n_a^2a + n_b^2b}{a+b}} \end{aligned}$$

The dispersion relation is approximately periodic when ω is approximately a common multiple of $\frac{n_a a}{c}$ and $\frac{n_b b}{c}$, so we expect a linear asymptote of slope equal to the low-frequency refractive index, as in Fig. 2.26. The mid-gap frequency of the first gap can thus be estimated to be

$$n_{\text{eff}}\frac{\pi}{d} \approx \pi\sqrt{\frac{n_a^2a + n_b^2b}{(a+b)^3}}$$

Problem 1.14

The visibility as a function of path difference is equal to

$$\begin{aligned}\left|\gamma(d)\right| &= \left|\frac{\Gamma(d)}{I_0}\right| \\ &= \left|\frac{\left\langle f(t)f(t-\frac{d}{c})\right\rangle}{I_0}\right|\end{aligned}$$

Where $f(t)$ is the superposition of two oppositely doppler shifted radiations, which has Gaussian power spectra

$$\begin{aligned}P(\omega) &= P_0(\omega) * \left[\delta(\omega+\Delta\omega)+\delta(\omega-\Delta\omega)\right] \\ P_0(\omega) &= C\exp\left(-\frac{(\omega-\omega_0)^2}{2\sigma^2}\right)\end{aligned}$$

Using Wienner-Khinchin theorem,

$$\begin{aligned}\gamma(d) = \gamma(\tau c) &\propto \mathcal{F}^{-1}[P(\omega)] \\ \gamma(\tau c) &\propto \mathcal{F}^{-1}[P_0(\omega)]\cos(\Delta\omega\tau)\end{aligned}$$

The inverse fourier transform of a Gaussian will be another Gaussian of width $\frac{c}{\sigma}$. Therefore, the form of the visibility curve is the absolute value of a Gaussian multiplied by a cosine. The velocity of expansion can be estimated from doppler equation (at low velocities)

$$\begin{aligned}\Delta\omega &= \frac{\omega_0 v}{c} \\ \cos\left(\Delta\omega\frac{d}{c}\right) &= 0 \quad \text{at } \Delta\omega = \frac{\pi c}{2d} \\ v &= \frac{c\lambda}{4d} \\ &= 24.6\,\mathrm{km\,s^{-1}}\end{aligned}$$

The apparent line width is

$$\delta\omega = 2.36\sigma = \frac{2.36\sqrt{2}c}{l_c}$$

where l_c is the coherent length, the path difference at which the visibility, *unmodulated by cosine*, drops to $\frac{1}{e}$ maximum.

$$l_c \approx 5.5\,\mathrm{mm}$$

so we have

$$\delta\omega \approx 1.81 \times 10^{11} \text{ rad s}^{-1}$$

If the linewidth is due to thermal broadening, the temperature of the hydrogen gas shell is

$$\sigma = \omega_0 \left(\frac{k_B T}{mc^2}\right)^{\frac{1}{2}}$$

$$T = \left(\frac{\sigma}{\omega_0}\right)^2 \frac{mc^2}{k_B}$$

$$T = 7759\,\text{K}$$

Problem 1.15

(a)

Coherence length is related to line width by

$$\delta\omega = \frac{2\pi c}{\lambda^2}\delta\lambda$$

$$l_c = \frac{2.36\sqrt{2}c}{\delta\omega}$$

$$l_c = \frac{2.36\sqrt{2}\lambda^2}{2\pi\delta\lambda}$$

$$l_c = 0.16\,\text{m}$$

(b)

The visibility of the fringes can be modelled to be

$$\gamma(d) = e^{-\frac{\sigma^2(\overbrace{2d}^{\text{return trip}})^2}{2c^2}} = \exp\left(-\frac{4d^2}{l_c^2}\right)$$

So when one of the mirrors is moved by 10 mm and 50 mm, the visibilityof the fringes decreases to

$$\gamma = 0.984 \qquad \text{and} \qquad \gamma = 0.677$$

respectively.

(c)

If the power spectrum is a top-hat of width $\Delta\omega$ centered at ω_0, the visibility function would be

$$\gamma(d) \propto \text{sinc}\left(\frac{d\Delta\omega}{c}\right)$$

which first falls to 0 at

$$\frac{d\Delta\omega}{c} = \pi \implies d = \frac{\lambda^2}{2\delta\lambda} = 0.16 \text{ m}$$

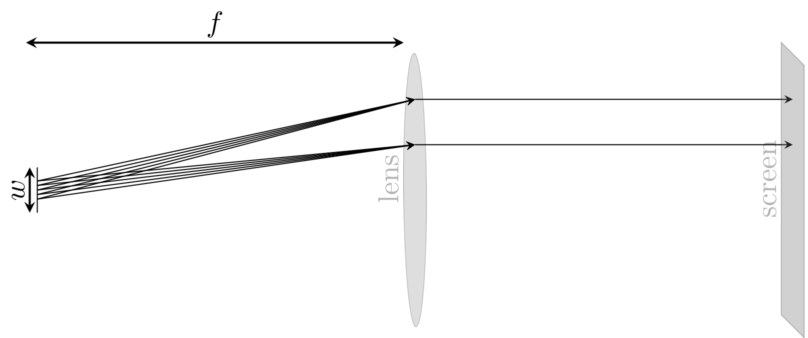
Problem 1.16

The width of the wire is small compared to the focal length, so we make several approximations:

$$\theta = \arcsin\left(\frac{x}{f}\right) = \frac{x}{f} \qquad (\alpha = \frac{w}{f})$$

$$\text{the angular intensity } I(\theta) = \frac{I_0}{\alpha}$$

$$u = kd$$



$$\begin{aligned}\gamma(u) &= \frac{1}{I_0}\int_{\frac{\alpha}{2}}^{\frac{\alpha}{2}} \frac{I_0}{\alpha} e^{iu\theta}\,\mathrm{d}\theta \\ &= \frac{2f\sin\left(\frac{kdw}{2f}\right)}{kdw} \\ &= \operatorname{sinc}\left(\frac{kdw}{2f}\right) \\ &= \operatorname{sinc}\left(\frac{\pi dw}{f\lambda}\right)\end{aligned}$$

For the degree of coherence to be 0, the smallest separation is when

$$\frac{\pi dw}{f\lambda} = \pi \implies d = \frac{f\lambda}{w} = 0.6\,\text{mm}$$

Topic 2

Problem 2.17

The gauge field **A** has the same symmetries as current density **J**. A dipole moment along Oz corresponds to a current loop whose only nonzero J component is constant along in ϕ. Therefore,

$$\nabla \times \mathbf{A} = \frac{1}{r^2 \sin\theta} \begin{pmatrix} \frac{\partial}{\partial\theta}\big(r \sin\theta A_\phi\big) \\ -r\frac{\partial}{\partial r}\big(r \sin\theta A_\phi\big) \\ 0 \end{pmatrix}$$

$$\nabla \times \mathbf{A} = \frac{1}{r^2 \sin\theta} \begin{pmatrix} \frac{\partial}{\partial\theta}\big(r \sin\theta A_\phi\big) \\ -r\frac{\partial}{\partial r}\big(r \sin\theta A_\phi\big) \\ 0 \end{pmatrix} = \frac{\mu_0 m}{4\pi r^3} \begin{pmatrix} 2\cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$

$$\begin{aligned} \frac{\partial}{\partial\theta}\big(\sin\theta A_\phi\big) &= \frac{\mu_0 m}{4\pi r^2} \sin(2\theta) \\ \sin\theta A_\phi &= -\frac{\mu_0 m}{4\pi r^2} \frac{\cos(2\theta)}{2} + f(r) \\ -\sin\theta \frac{\partial}{\partial r}\big(r A_\phi\big) &= \frac{\mu_0 m}{4\pi r^2} \sin^2(\theta) \\ r A_\phi &= \frac{\mu_0 m}{4\pi r} \sin(\theta) + g(\theta) \\ A_\phi &= \frac{\mu_0}{4\pi} \frac{m}{r^2} \sin\theta + \text{const.} \end{aligned}$$

Problem 2.18

Evaluate the divergence for the two fields.

1.

$$\frac{\partial}{\partial x_i} \frac{1}{r^3} = -\frac{3x_i}{r^5}$$

$$\begin{aligned} \nabla \cdot \mathbf{B} &= \frac{B_0 b}{r^3} \left[-\frac{3x(x-y)z}{r^2} + z - \frac{3y(x-y)z}{r^2} - z - \frac{3z(x^2-y^2)}{r^2} \right] \\ &= -\frac{B_0 b}{r^3} \left[\frac{6z(x^2-y^2)}{r^2} \right] \end{aligned}$$

which is nonzero and therefore disobeys Maxwell's equations.

2.

$$\begin{aligned}\nabla\cdot\mathbf{B} &= B_0b^2\left[\frac{z\frac{\partial r^2}{\partial r}}{(b^2+z^2)^2r}-\frac{2z}{(b^2+z^2)^2}\right]\\ &= 0\end{aligned}$$

This field is plausible because it obeys Maxwell's equations.

For the field in (b)

$$\begin{aligned}\mu_0\mathbf{J} &= \nabla\times\mathbf{B}\\ &= B_0b^2\begin{pmatrix}0\\ \frac{r}{(b^2+z^2)^2}-\frac{4z^2r}{(b^2+z^2)^3}\\ 0\end{pmatrix}\\ &= \frac{B_0b^2r(b^2-3z^2)}{(b^2+z^2)^3}\hat{\boldsymbol{\phi}}\\ \nabla\times\mathbf{A} &= \mathbf{B}\\ \begin{pmatrix}-\frac{\partial A_\phi}{\partial z}\\ 0\\ \frac{1}{r}\frac{\partial(rA_\phi)}{\partial r}\end{pmatrix} &= B_0b^2\begin{pmatrix}\frac{zr}{(b^2+z^2)^2}\\ 0\\ \frac{1}{b^2+z^2}\end{pmatrix}\\ \mathbf{A} &= \frac{B_0b^2r}{2(b^2+z^2)}\hat{\boldsymbol{\phi}}+\text{const.}\end{aligned}$$

Problem 2.19

Probe the direction of polarisation of **E** field at a point **r** from the box. Then, move in the direction $\mathbf{r}\times\mathbf{E}$.

- If the electric field strength does not change along this direction, the dipole is electric. Move along the direction of **E** to find the plane which maximises field strength or radiation power. The normal to the plane will be the direction of the dipole (and also the direction of polarisation at that point).
- If the electric field strength did change, the dipole is magnetic. Move along $\mathbf{r}\times\mathbf{E}$, find the plane at which radiation is maximised, and the direction of magnetic dipole is the normal of the plane or $\parallel \mathbf{r}\times\mathbf{E}$.

Assuming the length sacle of the antenna is the same as that of the box, the power dependence on the current for electric and magnetic dipoles can be calculated as

$$\langle P_E\rangle = \frac{\omega^4(\frac{\langle I_0\rangle d}{\omega})^2}{12\pi\epsilon_0c^3} \qquad \langle P_M\rangle = \frac{\mu_0\omega^4(I_0d^2)^2}{12\pi c^3}$$

respectively. If the power supplies are replaced with a matched load, the box can still be illuminated with microwaves, and the reradiated (scattered) signal will have the same patterns as emitted signal.

Problem 2.20

The farfield expansion of the vector potential corresponding to magnetic dipole is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-ik)\int \mathbf{J}(\mathbf{r}')(\hat{\mathbf{r}}\cdot\mathbf{r}')\,\mathrm{d}V$$
$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{ike^{ikr}}{r}\hat{\mathbf{r}}\times\mathbf{m}$$

where $\mathbf{m}$ is the magnetic dipole moment, and $\hat{\mathbf{r}}$ is the unit vector parallel to $\mathbf{r}$.

For a dipole in the x-y plane rotating about the z axis at angular frequency ω, working in cylindrical coordinates, we have

$$\mathbf{m} = \frac{m_0}{\sqrt{2}}\begin{pmatrix} e^{i(\omega t+\phi_0+\phi)} \\ ie^{i(\omega t+\phi_0+\phi)} \\ 0 \end{pmatrix}$$

$$\begin{aligned}
\mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi}ike^{ikr}\frac{m_0}{\sqrt{2}}\begin{pmatrix} \frac{\rho}{r} \\ 0 \\ \frac{z}{r} \end{pmatrix} \times \begin{pmatrix} e^{i(\omega t+\phi_0+\phi)} \\ ie^{i(\omega t+\phi_0+\phi)} \\ 0 \end{pmatrix} \\
&= \frac{\mu_0}{4\pi}ike^{ikr}\frac{m_0}{\sqrt{2}}\begin{pmatrix} -\frac{z}{r}ie^{i(\omega t+\phi_0+\phi)} \\ \frac{z}{r}e^{i(\omega t+\phi_0+\phi)} \\ \frac{\rho}{r}ie^{i(\omega t+\phi_0+\phi)} \end{pmatrix} \\
&= \frac{\mu_0}{4\pi}\frac{ike^{ikr}}{r}\frac{m_0}{\sqrt{2}}e^{i(\omega t+\phi_0+\phi)}\begin{pmatrix} -iz \\ z \\ i\rho \end{pmatrix}
\end{aligned}$$

$$\mathbf{B} = -ik\mathbf{A}\times\mathbf{n}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{k^2e^{ikr}}{r^2}\frac{m_0}{\sqrt{2}}e^{i(\omega t+\phi_0+\phi)}\begin{pmatrix} -iz \\ z \\ i\rho \end{pmatrix} \times \begin{pmatrix} \rho \\ 0 \\ z \end{pmatrix}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{k^2e^{ikr}}{r^2}\frac{m_0}{\sqrt{2}}e^{i(\omega t+\phi_0+\phi)}\begin{pmatrix} z^2 \\ -\rho^2+iz^2 \\ -z\rho \end{pmatrix}$$

1. In the x-y plane

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{k^2 e^{ikr}}{r^2}\frac{m_0}{\sqrt{2}}e^{i(\omega t+\phi_0+\phi)}(-\rho^2\hat{\phi})$$
$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}$$
$$\mathbf{E} = -i\frac{\mu_0}{4\pi}\frac{\omega k e^{ikr}}{r}\frac{m_0}{\sqrt{2}}e^{i(\omega t+\phi_0+\phi)}\rho\hat{\mathbf{z}}$$

   The radiation pattern is circular and the polarisation of the electric field is parallel to Oz.

2. The direction of polarisation, at an angle θ to Oz, is parallel to the vector

$$\begin{aligned} & -\cos(\theta)\hat{\boldsymbol{\rho}} - i\cos(\theta)\hat{\boldsymbol{\phi}} + \sin(\theta)\hat{\mathbf{z}} \\ =& \big[-\cos(\theta)\sin(\theta) + \sin(\theta)\cos(\theta)\big]\hat{\mathbf{r}} - i\cos(\theta)\hat{\boldsymbol{\phi}} + \big[-\cos^2(\theta) - \sin^2(\theta)\big]\hat{\boldsymbol{\theta}} \\ =& -i\cos(\theta)\hat{\boldsymbol{\phi}} - \hat{\boldsymbol{\theta}} \end{aligned}$$

   which is always perpendicular to $\hat{\mathbf{r}}$, the direction of outward propagation of the wave.

Problem 2.21

Assuming one of the principal axis of rotation of the magnet is parallel to z-axis, the rotational kinetic energy at any instant is given by

$$E_k = \frac{1}{2}I\omega^2$$

The formula for radiation loss for an oscillating magnetic dipole is

$$\langle P_M \rangle = \frac{\mu_0 M^2}{12\pi c^3}\omega^4$$

The rotating magnetic dipole $\mathbf{M}$ can be decomposed into two orthogonal dipoles oscillating in phase quadrature, which contribute to the total power independently.

$$P_{\mathbf{M}} = 2\langle P_M \rangle = \frac{\mu_0 M^2}{6\pi c^3}\omega^4 = C\omega^4$$

For $\dot{\omega} \ll \omega^2$, kinetic energy changes over a time scale much greater than the period of rotation, such that the instantaneous power can be approximated by the average over a period

$$P_{\mathbf{M}} = \frac{\mathrm{d}E_k}{\mathrm{d}t}$$

$$
\begin{aligned}
C\omega^4 &= I\omega\dot{\omega} \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{C}{I}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{\omega}}{\omega^3}\right) \\
0 &= \frac{\ddot{\omega}\omega^3 - 3\omega^2\dot{\omega}^2}{\omega^6} \\
&\boxed{\ddot{\omega}\omega = 3(\dot{\omega})^2}
\end{aligned}
$$

For the pulsar in the Crab Nebula, the period $T = 33$ ms and $\dot{T} = 36$ ns/day. Assume that it is a sphere (of uniform density) with radius 7 km and a mass equal to that of the Sun (2×10^{30} kg). We have

$$
\begin{aligned}
\omega &= \frac{2\pi}{T} \\
\dot{\omega} &= \frac{2\pi\dot{T}}{T^2}
\end{aligned}
$$

Substituting in

$$
\begin{aligned}
\frac{\mu_0 M^2}{6\pi c^3}\omega^4 &= \frac{2}{5} m_s R^2 \omega\dot{\omega} \\
M &= \sqrt{\frac{12 m_s R^2 \dot{\omega}\pi c^3}{5\mu_0\omega^3}} \\
M &= 2 \times 10^{27} \text{ A m}^2
\end{aligned}
$$

Near the equator the magnetic field is azimuthal,

$$
\begin{aligned}
B_\theta &= \frac{\mu_0 M}{4\pi R^3} \\
B_\theta &= 7 \times 10^8 \text{ T}
\end{aligned}
$$

Problem 2.22

Radiation resistance is the effective resistance R_r of an antenna, such that

$$
\langle P \rangle \equiv \left\langle I^2 \right\rangle R_r
$$

The *power gain* of an antenna is the angular distribution of time-averaged radial Poynting flux, normalised to 4π

$$
G(\theta, \phi) = \frac{4\pi N(\theta, \phi)}{\iint N(\theta, \phi) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi}
$$

For a plane wireloop of area a^2, the time-averaged radial Poynting flux is

$$\begin{aligned} N(\theta,\phi) &= \left\langle r^2 \hat{\mathbf{n}} \cdot (\mathbf{E} \times \mathbf{H}) \right\rangle (\theta,\phi) \\ &= \left\langle r^2 \frac{Z_0 k^2}{4\pi} e^{ikr} \sin\theta I a^2 \frac{k^2}{4\pi} e^{ikr} \sin\theta I a^2 \right\rangle \\ &= \frac{Z_0 \omega^4 \sin^2\theta}{16\pi^2 c^4} a^4 \left\langle I^2 \right\rangle \end{aligned}$$

Normalising the essential angular distribution, we get

$$\begin{aligned} 4\pi = \iint G(\theta,\phi) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi &= A \iint \sin^3\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \\ 2 &= A \int_0^\pi \sin^3\theta \, \mathrm{d}\theta \\ 2 &= A \left[-\cos\theta + \frac{\cos^3\theta}{3} \right]_0^\pi \\ G(\theta,\phi) &= \frac{3}{2} \sin^2\theta \end{aligned}$$

Integrating over the unit sphere, we get

$$\begin{aligned} \langle P \rangle &= \frac{8\pi}{3} \frac{Z_0 \omega^4}{16\pi^2 c^4} a^4 \left\langle I^2 \right\rangle \\ \langle P \rangle &= \frac{Z_0 \omega^4}{6\pi c^4} a^4 \left\langle I^2 \right\rangle \\ R_r &= \frac{\mu_0 \omega^4}{6\pi c^3} a^4 \end{aligned}$$

By conservation of energy, the cross-section of combined scattering and absorption is the total power per incident electromagnetic flux density

$$\begin{aligned} \sigma &= \frac{\langle P \rangle}{\langle N_{\text{inc}} \rangle} \\ &= \frac{\left\langle V^2 \right\rangle / 2R_r}{\left\langle B^2 \right\rangle c/\mu_0} \\ &= \frac{\left\langle (\omega B a^2)^2 \right\rangle / 2R_r}{\left\langle B^2 \right\rangle c^3 \varepsilon_0} \\ &= \frac{\omega^2 a^4}{c^3 \varepsilon_0} \frac{3\pi c^3}{\mu_0 \omega^4 a^4} \\ &= \frac{3\pi c^2}{\omega^2} \end{aligned}$$

Problem 2.23

For our purposes, the Earth can be estimated to be flat. The mass of gas above unit area of earth relates to atmospheric pressure

$$\frac{m}{A} = \frac{p_{\text{atm}}}{g}$$

The atmosphere can be modelled as an ideal gas of volumeric composition $\frac{1}{5}$ Oxygen and $\frac{4}{5}$ Nitrogen, which gives number per area

$$\begin{aligned}\frac{m}{A} &= \left(\frac{m_{\text{O}_2}}{5} + \frac{4m_{\text{N}_2}}{5}\right)\frac{N}{A} \\ \frac{N}{A} &= 2 \times 10^{29}\ \text{m}^{-2}\end{aligned}$$

On the assumption that a molecule can be represented as a *perfectly conducting sphere* with polarisability $\alpha = 4\pi r^3 \varepsilon_0$ of radius 0.1 nm, the phase change of ultraviolet radiation with wavelength $\lambda = 320$ nm over individual molecules can be neglected. We are therefore in Rayleigh scattering regime. Ignoring multiple scattering

$$\begin{aligned}\frac{\langle P\rangle_{\text{sc}}}{\langle P\rangle_{\text{in}}} &= \frac{\sigma_{\text{sc}}\langle N\rangle}{A\langle N\rangle} \\ &= \frac{1}{A}\frac{\mu_0^2\omega^4\alpha^2 N}{6\pi} \\ &= \frac{\mu_0^2\varepsilon_0^2(2\pi c)^4 16\pi^2 r^6}{6\pi\lambda^4}\frac{N}{A} \\ &= \frac{2^8\pi^5 r^6}{6\lambda^4}\frac{N}{A} \\ &\approx 25\%\end{aligned}$$

The molecules are randomly directed, so half of the re-radiated light are scattered away from earth, giving

$$\frac{\langle P\rangle_{\text{lost}}}{\langle P\rangle_{\text{in}}} = \frac{25\%}{2} = 13\%$$

Problem 2.24

Cambridge is 52° N. Assume the Sun, a point (far) source, is coplanar with the equator of the Earth on March 21. At noon, Cambridge is nearest the Sun, sunlight, arriving at the atmosphere parallel to the equator, is scattered by the piece of atmosphere above Cambridge through an angle $\alpha = 52°$. The degree of polarisation is therefore

$$\begin{aligned}P &= \frac{1-\cos^2\alpha}{1+\cos^2\alpha} \\ P &= 45\%\end{aligned}$$