PNP Examples

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Contents

Topic 1 Matter and forces

Problem 1 Classification of particles

Generally, elementary particles can be classified into *bosons*, which have integer spin, and *fermions*, which have half integer spin.

In the Standard Model, all matter is made up of fundamental Fermions, which consist of three generations (replica) and corresponding antiparticles of *leptons* and *quarks*. Lepton particles include electron, muon, and tauon, and three corresponding neutrinos. Quark particles include the up and down, the charm and strange, and the top and bottom quarks. Quarks do not exist as free particles, they are confined within *hadrons*, which consist either of a quark and an antiquark (meson), or of three quarks (baryon).

Force (Strong and Electro-Weak) is mediated by gauge *bosons*, which are vector bosons of spin 1. The gluons mediate strong force, the photon mediates electromagnetic force, and the W and Z bosons mediate weak force.

Topic 2 Relativistic kinematics

Problem 2 Natural units

Natural units in particle physics are defined by setting $\hbar = c = 1$, and choosing eV as the unit of measurement for energy, such that time, length, and mass follow from $[E] = [\frac{\hbar}{t}] = [\frac{\hbar c}{x}] = [mc^2]$.

(a)

For a pion, the reduce compton wavelength is

$$\begin{aligned}\bar{\lambda} &= \frac{1}{m_\pi} \\ &= 7.16 \times 10^{-3} \text{ MeV}^{-1} \\ &= \frac{\hbar c}{m_\pi c^2} \\ &= 1.41 \times 10^{0} \text{ fm}\end{aligned}$$

(b)

A cross-section is written in natural units

$$\sigma = \frac{4}{3}\frac{\pi\alpha^2}{s}$$

Given $s = m_Z^2$, $m_Z = 91.2 \times 10^3$ GeV

$$\sigma = 2.68 \times 10^{-14} \text{ MeV}^{-2}$$

$$
\begin{aligned}
&= \frac{4}{3}\frac{\pi\alpha^2\hbar^2c^2}{s^2} \\
&= 1.04 \times 10^{-39}\ \mathrm{m}^2 \\
&= 1.04 \times 10^{-11}\ \mathrm{m}^2
\end{aligned}
$$

Problem 3

A particle X decays into two particles a and b.

(a)

Mass-energy is conserved in the reaction. In rest frame of X, momenta of a and b are opposite and equal.

$$
\begin{aligned}
E_a + E_b &= m_X \\
p_a^2 &= p_b^2 \\
E_a^2 - m_a^2 &= E_b^2 - m_b^2 \\
E_a^2 &= m_X^2 + E_a^2 - 2m_X E_a + m_a^2 - m_b^2 \\
E_a &= \frac{m_X^2 + m_a^2 - m_b^2}{2m_X}
\end{aligned}
$$

Similarly particle b has

$$E_b = \frac{m_X^2 + m_b^2 - m_a^2}{2m_X}$$

If the two particles have identical mass (are the same or antiparticles of each other),

$$E_i = \frac{m_X}{2} \qquad i \in \{a, b\}$$

(b)

The momenta of a and b are both equal to

$$
\begin{aligned}
p_a^2 &= E_a^2 - m_a^2 \\
p_a^2 &= \frac{m_X^4 + m_a^4 + m_b^4 + 2m_X^2m_a^2 - 2m_X^2m_b^2 - 2m_a^2m_b^2}{4m_X^2} - m_a^2 \\
p_b = p_a &= \frac{\sqrt{m_X^4 + m_a^4 + m_b^4 - 2m_X^2m_a^2 - 2m_X^2m_b^2 - 2m_a^2m_b^2}}{2m_X}
\end{aligned}
$$

If the masses are the same

$$p_a = p_b = \frac{\sqrt{m_X^2 - 4m_a^2}}{2}$$

If particle b is massless

$$p = \frac{\sqrt{m_X^4 + m_a^4 - 2m_X^2m_a^2}}{2m_X} = \frac{m_X^2 - m_a^2}{2m_X}$$

(c) The HERA collider at DESY provided head-on collisions between an electron beam of 27.5 *GeV and a proton beam of* 920 *GeV.*

Let a be electron and b be proton. The oppositely-directed momenta of the proton and electron are

$$p_a^2 = 7.56 \times 10^8 \text{ MeV}^2$$
$$p_b^2 = 8.46 \times 10^{11} \text{ MeV}^2$$

The center-of-mass energy squared

$$s = (E_a + E_b)^2 - (\mathbf{p}_a + \mathbf{p}_b)^2$$
$$s = 2E_aE_b + 2p_ap_b + m_a^2 + m_b^2$$
$$s = 1.01 \times 10^{11} \text{ MeV}^2$$

is invariant. In the frame where the proton is at rest

$$s = \left(E_a' + m_b\right)^2 - {p_a'}^2$$
$$s = m_a^2 + m_b^2 + 2E_a'm_b$$
$$E_a' = \frac{s - m_a^2 - m_b^2}{2m_b}$$
$$E_a' = \frac{E_aE_b + p_ap_b}{m_b}$$
$$E_a' = 5.39 \times 10^7 \text{ MeV}$$

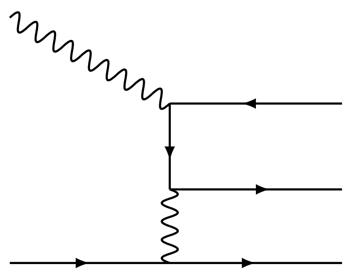
The Higgs boson of mass 125 GeV is less than $\sqrt{s}$, meaning HERA would have had sufficient energy to produce it.

Problem 4 Omega baryon decay

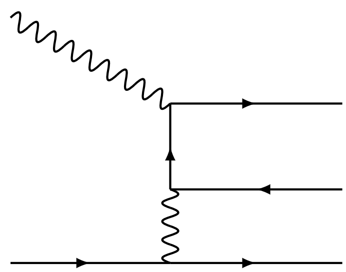
(a)

If the process $\gamma \to e^+e^-$ were possible, we would be able to describe this in the zero momentum frame of the product particles. However, in this frame, the momentum of photon would be zero, while its energy is nonzero. Therefore, the process is proven to be forbidden in vacuum by violation of $E = pc$.

In the presence of matter, this is possible because temporarily unconserved energy can be compensated by electromagnetic interaction with background charge. In a Feynman diagram, this is represented by a virtual electron/positron becoming real by exchanging a virtual photon with liquid Hydrogen.



and



The momentum of Ω^- squared is

$$\begin{aligned} p_\Omega^2 &= |\mathbf{p}_1 + \mathbf{p}_2|^2 \\ &= p_1^2 + p_2^2 + 2p_1 p_2 \cos(71^\circ) \\ &= 4.061 \times 10^6 \text{ MeV}^2 \\ p_\Omega &= 2.015 \times 10^3 \text{ MeV} \end{aligned}$$

Its energy is the sum of E_1 and E_2, so

$$\begin{aligned} E_\Omega &= \sqrt{m_1^2 + p_1^2} + \sqrt{m_2^2 + p_2^2} \\ E_\Omega^2 &= 6.914 \times 10^6 \text{ MeV}^2 \end{aligned}$$

Giving its mass

$$m_\Omega = \sqrt{E^2 - p^2} = 1.689 \times 10^3 \text{ MeV}$$

(b)

γ of the Ω particle is

$$\gamma = \frac{E}{m}$$

and its speed is

$$v = \frac{p}{\gamma m}$$

so its proper lifetime is

$$\begin{aligned} \tau &= \frac{l}{\gamma v} \\ &= \frac{lm}{p} \\ &= 6.99 \times 10^{-11} \text{ s} \end{aligned}$$

Topic 3 Decays and reactions

Problem 5 Radioactive decay

N.B. In this question τs are mean-lifes.

(a)

Bt time t, the proportion of ^{198}Au that were produced at t' which is left is

$$e^{-\frac{1}{\tau}(t-t')}$$

so given a reaction rate of $R = 10^{10}\ s^{-1}$, the total number N of ^{198}Au atoms after $t = 6$ days is

$$\begin{aligned} N &= \int_0^t e^{-\frac{1}{\tau}(t-t')} R\,\mathrm{d}t' \\ &= 8.64 \times 10^{14} \times e^{-6/4} \times 4 \times \left(e^{6/4} - 1\right) \\ &= 2.68 \times 10^{15} \end{aligned}$$

(b)

The number N_{Hg} is equal to the number of reacted ^{198}Au atoms, which is given by

$$\begin{aligned} N_{\mathrm{Hg}} &= Rt - N \\ &= 2.50 \times 10^{15} \end{aligned}$$

(c)

The equilibrium number ^{198}Au is

$$N_{eq} = \lim_{t\to\infty} 8.64 \times 10^{14} \times e^{-t/4} \times 4 \times \left(e^{t/4} - 1\right) = 8.64 \times 10^{14} \times 4 = 3.456 \times 10^{15}$$

Problem 6 Caesium decay

N.B. In this question τs are half-lifes.

The rate equations

$$\begin{aligned} \frac{\mathrm{d}N_{Cs}}{\mathrm{d}t} &= -\lambda_{Cs} N_{Cs} \\ \frac{\mathrm{d}N_{Ba}}{\mathrm{d}t} &= -\frac{\mathrm{d}N_{Cs}}{\mathrm{d}t} - \lambda_{Ba} N_{Ba} \\ \frac{\mathrm{d}N_{La}}{\mathrm{d}t} &= \lambda_{Ba} N_{Ba} \end{aligned}$$

the maths are analogous to 1.5

$$N_{Ba}(t) = \int_0^t \exp\left(-\lambda_{Ba}(t-t')\right) \lambda_{Cs} N_{Cs}(0) \exp\left(-\lambda_{Cs} t'\right) \mathrm{d}t'$$

$$N_{Ba}(t) = \lambda_{Cs} N_{Cs}(0) \exp(-\lambda_{Ba} t) \int_0^t \exp\left[+(\lambda_{Ba} - \lambda_{Cs}) t'\right] \mathrm{d}t'$$
$$N_{Ba}(t) = \frac{\lambda_{Cs} N_{Cs}(0)}{\lambda_{Ba} - \lambda_{Cs}} \left[\exp(-\lambda_{Cs} t) - \exp(-\lambda_{Ba} t)\right]$$

Maximum activity occurs when N_{Ba} is maximum, which is when

$$\lambda_{Cs} \exp(-\lambda_{Cs} t) - \lambda_{Ba} \exp(-\lambda_{Ba} t) = 0$$
$$t = \frac{1}{\lambda_{Ba} - \lambda_{Cs}} \ln\left(\frac{\lambda_{Ba}}{\lambda_{Cs}}\right) = -\frac{1}{\ln(2)/\tau_{Ba} - \ln(2)/\tau_{Cs}} \ln\left(\frac{\tau_{Ba}}{\tau_{Cs}}\right) = 33.5 \text{ min}$$
$$A_{Ba} = \lambda_{Ba} N_{Ba} = \lambda_{Cs} N_{Cs}(0) \left(\frac{\lambda_{Cs}}{\lambda_{Ba}}\right)^{\frac{\lambda_2}{\lambda_1 - \lambda_2}} = 0.0866 \text{ mCi}$$

Problem 7 Kaon decay

N.B. In this question τs are mean-lifes.

(a)

Use $\hbar = 6.6 \times 10^{-16}$ eV s, the total width of K^+ is

$$\frac{\hbar}{\tau} = 5.5 \times 10^{-8} \text{ GeV}$$

and the branching ratio of the particular decay with partial width 1.2×10^{-8} GeV

$$\frac{1.2}{5.5} = 21.8\%$$

(b)

Mass of Kaon is 493 MeV, so the fraction which hasn't decayed is

$$\exp\left(-\frac{m \Delta x}{p \tau}\right) = 0.25$$

(c)

In the zero momentum frame of the decay, the momentum of π^+ can be quoted from problem 1.3

$$2 \times \sqrt{m_\pi^2 + p_{\pi(0)}^2} = m_{K^+}$$
$$p_{\pi(0)} = \frac{\sqrt{m_{K^+}^4 + m_{\pi^+}^4 + m_{\pi^0}^4 - 2m_{K^+}^2 m_{\pi^+}^2 - 2m_{K^+}^2 m_{\pi^0}^2 - 2m_{\pi^+}^2 m_{\pi^0}^2}}{2m_{K^+}}$$

$$E_{\pi(0)} = \sqrt{p_{\pi(0)}^2 + m_{\pi(0)}}$$

The zero momentum frame and the lab frame are related by

$$\beta = \frac{p}{E} = \sqrt{\frac{1}{1 + \frac{m^2}{p^2}}}$$

and

$$\gamma = \sqrt{\frac{1}{1-\beta^2}} = \sqrt{\frac{p^2+m^2}{m^2}}$$

where p and E are momentum and energy of the Kaon. The ambiguity of the energy of pion in lab frames arises from the arbitrary direction of p_π in the ZMF. The maximum and minimum values occur when p_π is collinear with p.

$$\max(E_\pi) = \gamma E_{\pi(0)} + \gamma\beta p_\pi = 9.19 \times 10^3 \text{ MeV}$$
$$\min(E_\pi) = \gamma E_{\pi(0)} - \gamma\beta p_\pi = 0.88 \times 10^3 \text{ MeV}$$

Problem 8 Cross-sections

Total cross-section of a state is the sum of all process cross-section from the state to another. Differential cross-section is the angular distribution of the effective target area.

(a) transmission

The nonelastic cross-section is

$$\sigma_{ne} = 270 \times 10^{-28} \text{ m}^2$$

the total number of elastically scattered and non-scattered neutrons is

$$\begin{aligned} N_t &= I(1 - \sigma_{ne} d_{(\text{thickness})} \frac{1}{m_{^{235}\text{U}}}) \\ &= 9.93 \times 10^4 \text{ s}^{-1} \end{aligned}$$

(b) fission

$$\begin{aligned} N_{fi} &= I \sigma_{fi} d \frac{1}{m_{^{235}\text{U}}} \\ &= 5.12 \times 10^2 \text{ s}^{-1} \end{aligned}$$

(c) elastic scattering

$$\begin{aligned} \Phi_e &= \frac{N_e}{4\pi R^2} = I\sigma_e d \frac{1}{m_{^{235}\text{U}}} \\ &= 2.04 \times 10^{-5} \text{ m}^{-2} \text{ s}^{-1} \end{aligned}$$

Problem 9 Breit-Wigner formula

The Breit-Wigner formula for a reaction cross-section is given by

$$\sigma(E) = \frac{\pi g}{p_i^2} \frac{\Gamma_{Z\to i}\Gamma_{Z\to f}}{(E-E_0)^2 + \Gamma^2/4}$$

- p_i is the momentum of the incoming particles in zero momentum frame.
- E is the total energy of colliding particles in zero momentum frame.
- E_0 is the energy of the intermediate state Z.
- $\Gamma_{Z\to i,f}$ are the partial decay rates from Z to i, f
- Γ is the total decay rate of Z.
- g is the fraction of spin of i that coincides with spin Z over the total number of spin states of i.

The Breit-Wigner formula is the product of the density of states and the modulus squared of the matrix element of the transition, which is derived from second order perturbation theory. The damping term in the Lorentzian comes from the decaying probability of the Z state.

For elastic process, both i and f are $n + {}^{123}\text{Te}$,

$$\begin{aligned}
\sigma_n(E) &= \frac{\pi g}{p_i^2} \frac{\Gamma_n^2}{(E-E_0)^2 + \Gamma^2/4} \\
&= \frac{\Gamma_n}{\Gamma_\gamma} \frac{\pi g}{p_i^2} \frac{\Gamma_n \Gamma_\gamma}{(E-E_0)^2 + \Gamma^2/4} \\
&= \frac{\Gamma_n}{\Gamma_\gamma} \sigma_\gamma(E) \\
&= 7.4 \times 10^3 \text{ b} \qquad \text{at resonance}
\end{aligned}$$

Given "neutron energy" = 2.2 eV much less than the rest mass energy, assuming this is the non-relativistic kinetic energy in ZMF

$$p_i = \sqrt{2 m_n E_n^{\text{nonrelativistic}}} = 6.43 \times 10^4 \text{ eV}$$

substituting into Breit-Wigner formula

$$\begin{aligned}
\sigma_\gamma(E) &= \frac{\pi g}{p_i^2} \frac{\Gamma_n \Gamma_\gamma}{(E-E_0)^2 + \Gamma^2/4} \\
g &= \left(\frac{\pi}{\sigma_\gamma^{\text{resonance}} p_i^2} \frac{\Gamma_n \Gamma_\gamma}{\Gamma^2/4} \right)^{-1}
\end{aligned}$$

$$g = 0.78$$
$$J_{^{124}\mathrm{Te}} = \frac{1}{2}\left[g(2J_{^{123}\mathrm{Te}} + 1)(2J_n + 1) - 1\right]$$
$$J_{^{124}\mathrm{Te}} = \frac{1}{2}[4g - 1]$$
$$J_{^{124}\mathrm{Te}} = 1.05 \approx 1$$

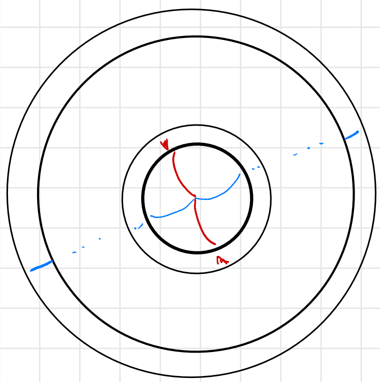
The spin of ^{124}Te is 1.

Topic 4 Colliders and detectors

Problem 10 Detector signatures

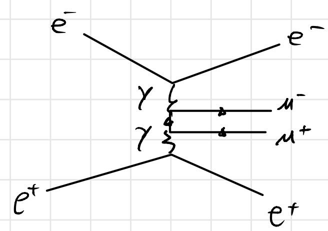
10.

a)

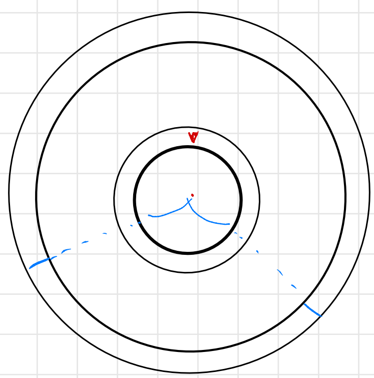


e^+e^- : track, EM deposit.

$\mu^+\mu^-$: track, penetrating

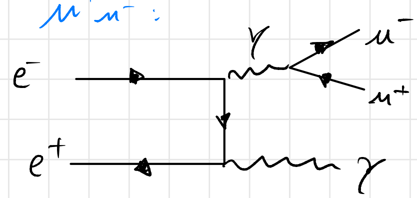


b)



γ: Momentum, EM deposit

$\mu^+\mu^-$:



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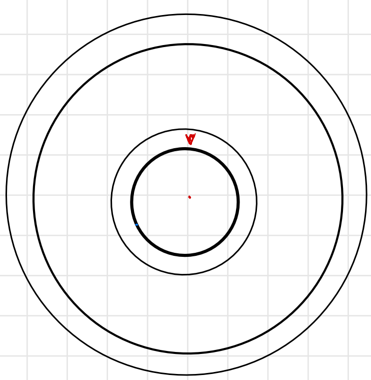
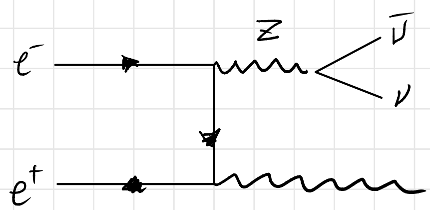
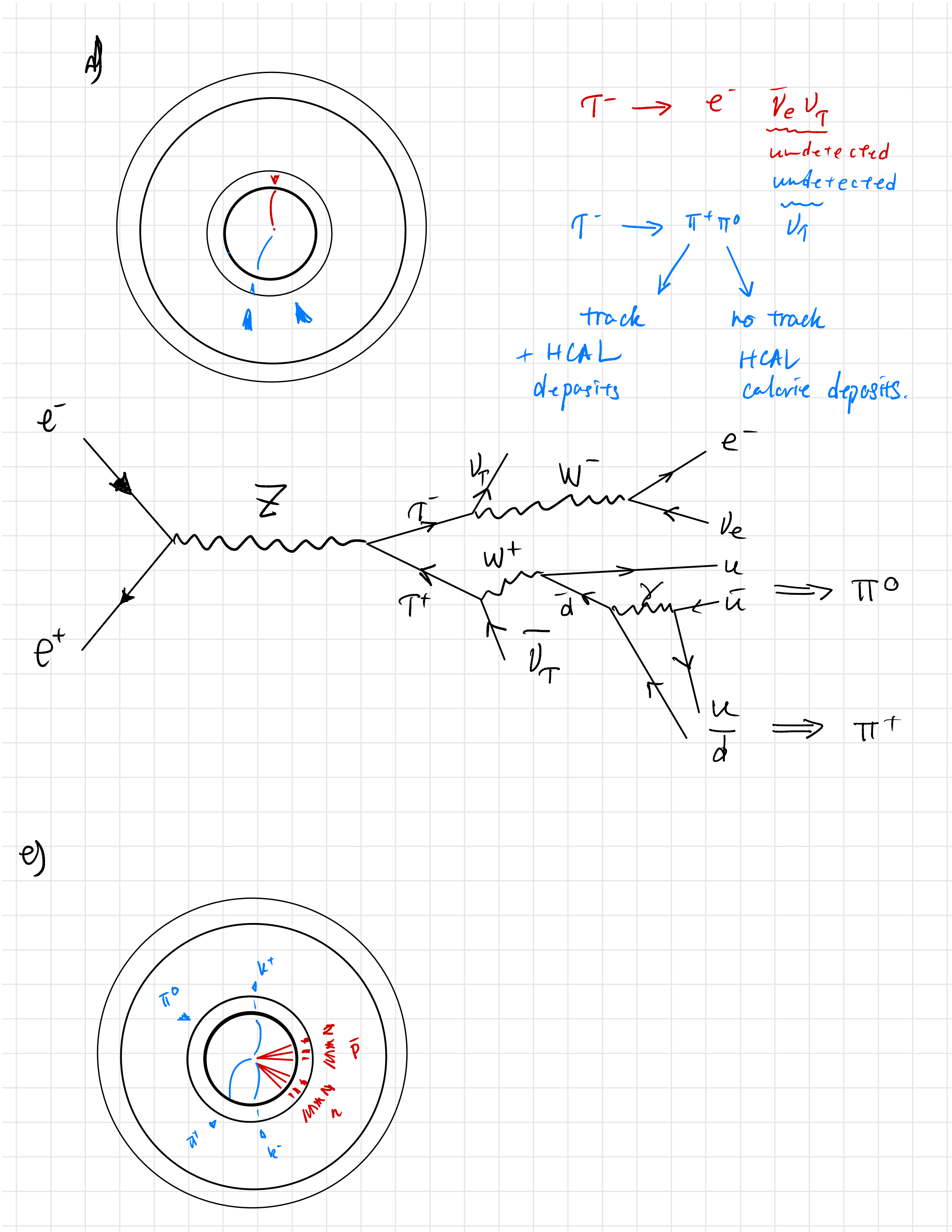


Neutrinos : not detected




d)
τ⁻ → e⁻ ν̄e ντ
undetected
undetected
ντ
τ⁻ → π⁺π⁰
track
+ HCAL
deposits
no track
HCAL
calorie deposits.
e⁻
Z
τ⁻
ντ
W⁻
e⁻
νe
e⁺
W⁺
τ⁺
u
ū ⟹ π⁰
d̄
γ
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u
d̄ ⟹ π⁺
e)
K⁺
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p̄
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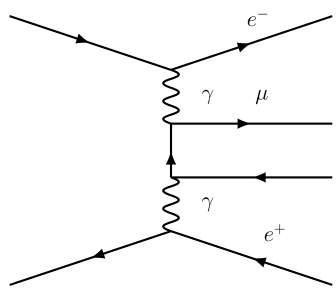
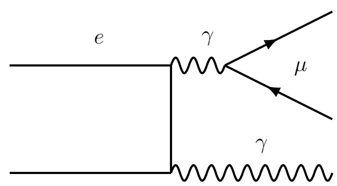
π⁺
K⁻

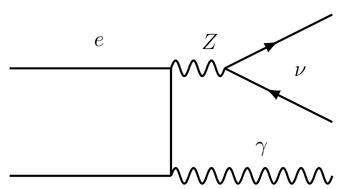
(a) $e^+e^- \to \mu^+\mu^-e^+e^-$



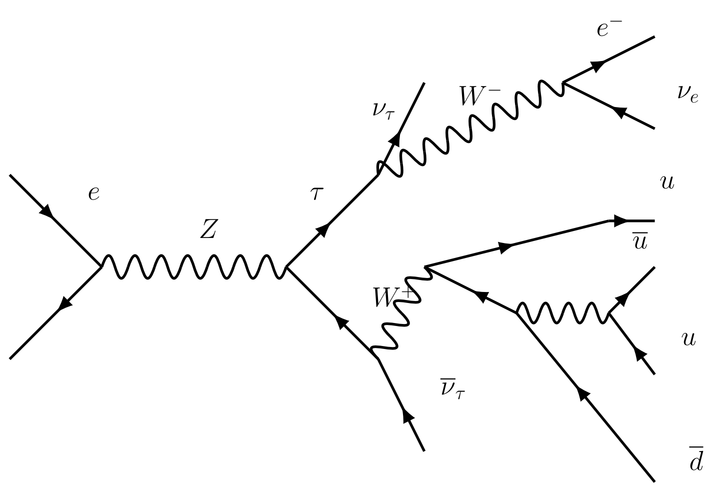
(b) $e^+e^- \to \mu^+\mu^-\gamma$



(c) $e^+e^- \to \nu\overline{\nu}\gamma$



(d) $e^+e^- \to \tau^+\tau^-$, $\tau^- \to e^-\nu_e\nu_\tau$, $\tau^+ \to \pi^0\pi^+\overline{\nu}_\tau$



Problem 11 Detector resolution

(a)

In the tracking detector

$$\frac{\sigma_p}{p} \propto p$$

Muon mass is 6 eV $\ll$ 1 GeV so $p \approx E$. If the momentum accuracy is 1% for 1 GeV muons, it is 20% for 20 GeV muons.

(b)

In the calorimeters (both ECAL and HCAL)

$$\frac{\sigma_E}{E} \propto E^{-1/2}$$

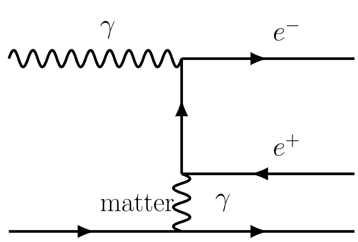
Given the energy resolution for 1 GeV electrons in the electromagnetic calorimeter is 0.5%, the energy resolution for 10 GeV electrons is

$$\frac{\sigma_E}{E} = 0.5\% \times 1^{1/2} \times 10^{-1/2} = 0.16\%$$

Topic 5 Feynman diagrams and QED

Problem 12 QED diagrams

(a)



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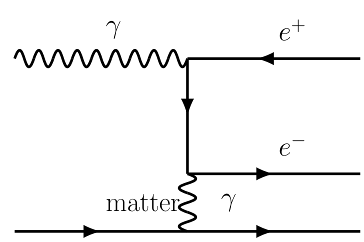

(b)

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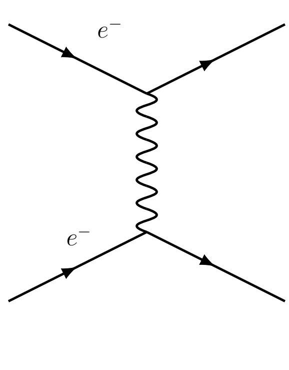

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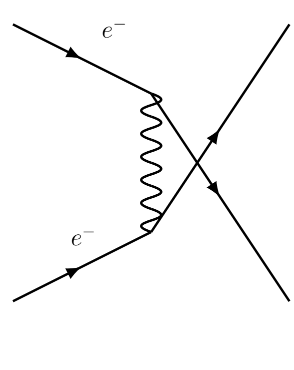
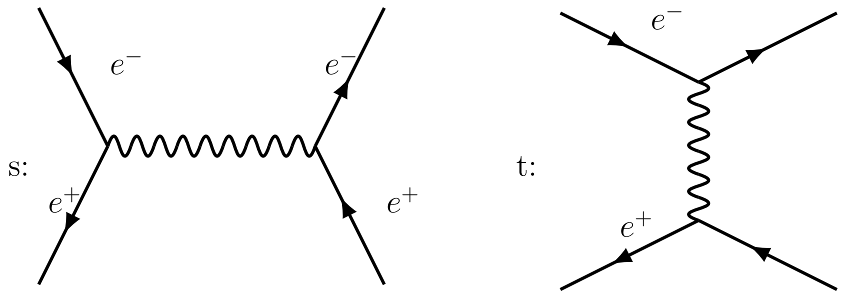
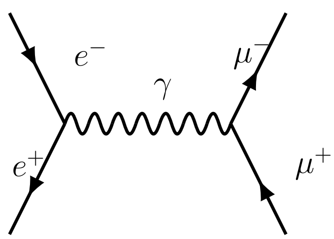


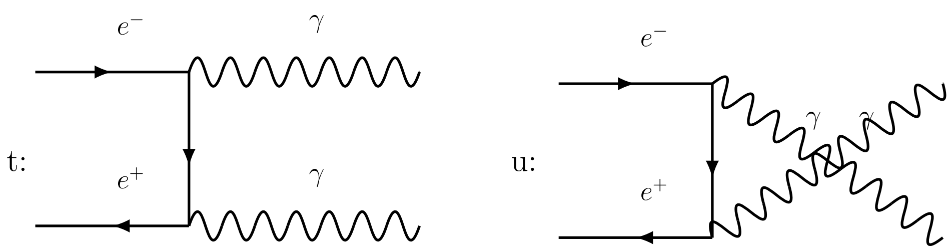
(c)



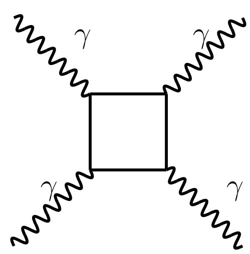
(d)



(e)



(f)

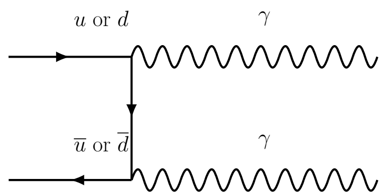


The middle square is in principle the sum over time orderings of any charged fermion, but is dominated by the electron which is the lightest.

Problem 13 Pion decay

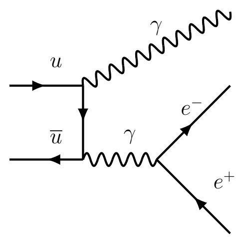
(a)

The π^0 predominantly decays to $\gamma\gamma$



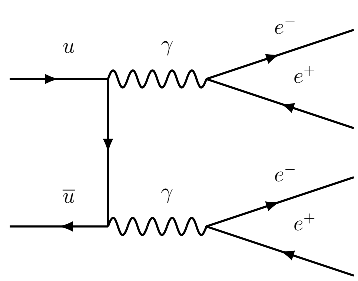
The coupling strength is $\left(Q_q\sqrt{\alpha}\right)^2 = Q_q^2\alpha$.

The pion also decays to $e^+e^-\gamma$ via (the $d\overline{d}$ pion is analogous)



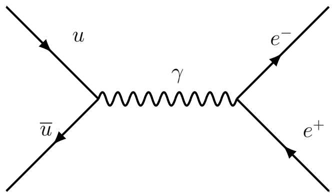
The coupling strength is $2Q_q^2\sqrt{\alpha}Q_q\sqrt{\alpha}\sqrt{\alpha} = 2Q_q^2\sqrt{\alpha^3}$. It gains a factor 2 because there are two photons that can pair-create.

Or, to $e^+e^-e^+e^-$ via

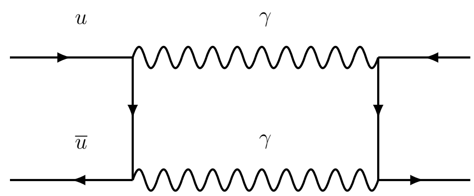


coupling strength $Q_q\sqrt{\alpha}Q_q\sqrt{\alpha}\sqrt{\alpha}\sqrt{\alpha} = Q_q^2\alpha^2$.

Or to e^+e^-



or



The left is the lowest order, but the meson which has $J^p = 0^-$ does not have the appropriate spin to annihilate to a photon which has spin ± 1.

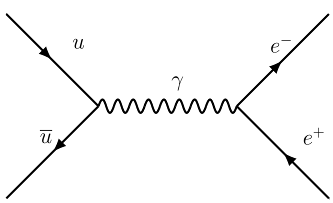
Ignoring propagator effects, the branching factors are roughly proportional to the coupling strengths squared.

$$\frac{\Gamma_{\gamma e^+e^-}}{\Gamma_{\gamma\gamma}} : \frac{\Gamma_{e^+e^-e^+e^-}}{\Gamma_{\gamma\gamma}} : \frac{\Gamma_{e^+e^-}}{\Gamma_{\gamma\gamma}} \approx \frac{4}{137} : \frac{1}{137^2} : \frac{1}{137^2} \approx 2.9\% : 5.3 \times 10^{-5} : 5.3 \times 10^{-5}$$

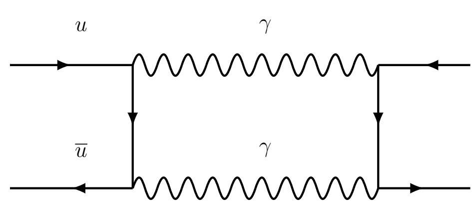
which **do not** coincide with 1.2%, 3.2×10^{-5}, and 2×10^{-7} respectively.

(b)

The ρ^0 meson is $\frac{1}{\sqrt{2}}\left(u\overline{u} - d\overline{d}\right)$, its decay to e^+ e^- are also, to the lowest orders



or



The ρ^0 has $J^p = 1^-$ so the left diagram is allowed. The expected ratio of partial widths to the two decays $\pi^0 \to e^+e^-$ and $\pi^0 \to e^+e^-$, based only on coupling strengths, is

$$\left(Q_q\sqrt{\alpha}Q_q\sqrt{\alpha}\sqrt{\alpha}\sqrt{\alpha}\right)^2 : \left(Q_q\sqrt{\alpha}\sqrt{\alpha}\right)^2 = 1 : 137^2 = 1 : 1.8 \times 10^4$$

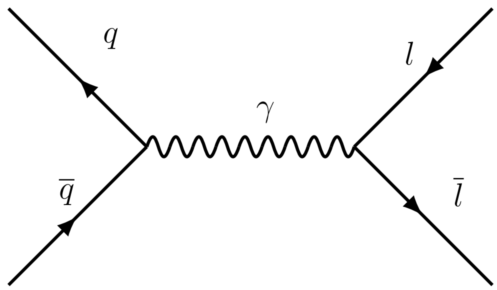
The real ratio between these partial widths are

$$\Gamma_{\pi^0\to e^+e^-} : \Gamma_{\rho^0\to e^+e^-} = \frac{2\times 10^{-7}}{8.4\times 10^{-17}}\hbar : \frac{4\times 10^{-5}}{4.4\times 10^{-24}}\hbar = 1.56\ \mu\text{eV} : 5.98\ \text{keV}$$

The ratios **do not** coincide.

Problem 14 Drell-Yan

The typical Drell-Yan process is the Feynman diagram



The following hadron-hadron interactions are compared for their cross-sections (the colours of the annihilating quark-antiquark pair must match so overall every coupling strength is suppressed by a factor of 3 which does not effect ratios)

1. The π^+ is bound state of $u\overline{d}$ and p is bound state of uud. The only quark-antiquark pair is d and $\overline{d}$. The cross-section is (proportional to)

$$(Q_d\sqrt{\alpha})^2 = \frac{1}{9}\alpha$$

2. The π^+ is bound state of $u\overline{d}$ and n is bound state of udd. The total Drell-Yan cross-section is the sum of two possible combinations of d and $\overline{d}$

$$(Q_d\sqrt{\alpha})^2 + (Q_d\sqrt{\alpha})^2 = \frac{2}{9}\alpha$$

3. The π^- is bound state of $d\overline{u}$ and p is bound state of uud. The total Drell-Yan cross-section is the sum of two possible combinations of d and $\overline{u}$

$$(Q_u\sqrt{\alpha})^2 + (Q_u\sqrt{\alpha})^2 = \frac{8}{9}\alpha$$

4. The π^- is bound state of $d\overline{u}$ and n is bound state of udd. The only quark-antiquark pair is u and $\overline{u}$. The cross-section is

$$(Q_u\sqrt{\alpha})^2 = \frac{4}{9}\alpha$$

The Drell-Yan cross-sections of these interactions are therefore expected to be in the ratio $1:2:8:4$.

pp is not expected to have any first order Drell-Yan interaction because no antiquark is bound in p. $\overline{p}p$ has 4 possible $u\overline{u}$ pairs and 1 $d\overline{d}$ pair, so it is expected to have cross-section proportional to

$$4 \times (Q_u\sqrt{\alpha})^2 + (Q_d\sqrt{\alpha})^2 = \frac{17}{9}\alpha$$

which is 17 times the Drell-Yan cross-section of π^{+p}.

Topic 6 QCD and the quark model

Problem 15 Spin and parity

Both strong and electromagnetic interactions conserve parity. The parity of the $l = 0$ pionic atom

$$P_\pi P_d (-1)^l = P_\pi P_d = P_\pi$$

is thence equal to the parity of the product

$$P_{nn} = P_n P_n (-1)^{l_{\text{end}}}$$

Given that the pion is spinless and has no angular momentum in s orbital, the Deuteron has spin 1, conserving angular momentum at the end requires

$$j = 1$$

Neutrons are spin-$\frac{1}{2}$ fermions so their end state is overall antisymmetric. If their spin parts are symmetric, their spatial part must be antisymmetric $\implies l_{\text{end}}$ is odd. If their spin parts are antisymmetric $\implies s = 0 \implies l_{\text{end}} = j = 1 \implies$ spatial part is also antisymmetric, which disobeys overall antisymmetry of fermions.

Hence the final state must have odd l_{end} and negative overall parity, and

$$P_\pi = P_{nn} = -1.$$

Problem 16

(a)

Using

$$M_{q\bar{q}} = m_1 + m_2 + A\frac{\mathbf{S}_1 \cdot \mathbf{S}_2}{m_1 m_2}$$

and

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = \frac{1}{2}S^2 - \frac{3}{4}$$

Meson	quark content	J^p	S^2	$\mathbf{S}_1 \cdot \mathbf{S}_2$	predicted mass / MeV
π	$u\overline{d}$	0^-	0	$-3/4$	140
K	$u\overline{s}$	0^-	0	$-3/4$	484
η	$\frac{1}{\sqrt{6}}\left(u\overline{u} + d\overline{d} - 2s\overline{s}\right)$	0^-	0	$-3/4$	559
ρ	$u\overline{d}$	1^-	2	$1/4$	780
ω	$\frac{1}{\sqrt{2}}\left(u\overline{u} + d\overline{d}\right)$	1^-	2	$1/4$	780
ω	$\frac{1}{\sqrt{2}}\left(u\overline{u} + d\overline{d}\right)$	1^-	2	$1/4$	780
K	$u\overline{s}$	1^-	2	$1/4$	896
ϕ	$s\overline{s}$	1^-	2	$1/4$	1032

where masses of particles with mixed quark contents are generalised to

$$M = \langle M \rangle = \sum_i P_i M_i$$

where P_i are the probabilities of a particular quark content state and M_i the corresponding mass.

For $\eta' = \frac{1}{\sqrt{3}}\left(u\overline{u} + d\overline{d} + s\overline{s}\right)$, $\mathbf{S}_1 \cdot \mathbf{S}_2 = -3/4$

$$M = \frac{1}{3}(140 + 140 + 768) = 349 \text{ MeV}$$

(way off from measured mass 958 MeV)

(b)

The total spin of any *pair* of quarks in the $J = \frac{3}{2}$ baryon decuplets must all be 1 so that the total spin quantum number adds up to $\frac{3}{2}$. With

$$M_{qqq} = m_1 + m_2 + m_3 + A' \frac{\mathbf{S}_1 \cdot \mathbf{S}_2}{m_1 m_2} + A' \frac{\mathbf{S}_1 \cdot \mathbf{S}_3}{m_1 m_3} + A' \frac{\mathbf{S}_2 \cdot \mathbf{S}_3}{m_2 m_3}$$

Baryon	quark content	predicted mass / MeV	measured mass / MeV
Δ	ddd	1230	1232
Σ	dds	1377	1385
Ξ	dss	1529	1533
Ω^-	sss	1687	1672

Problem 17 Proton and neutron moments

(a)

The total wavefunction of 3 indistinguishable quarks

$$\psi = \psi_{\text{space}}\psi_{\text{colour}}\psi_{\text{flavour}}\psi_{\text{spin}}$$

has to be antisymmetric. All the baryon ground states $l = 0$ have symmetric space components and antisymmetric colour components, so the flavour and spin parts combined have to be symmetric.

For the two u quarks in proton, the flavour part is already symmetrised so their spins are also symmetric, so

$$\psi_{uu} = u\uparrow u\uparrow \quad \text{or} \quad u\downarrow u\downarrow \quad \text{or} \quad \frac{1}{\sqrt{2}}(u\uparrow u\downarrow + u\downarrow u\uparrow)$$

The first two are evidently $|S, S_z\rangle = |1, \pm 1\rangle$, and the third is proportional to $|1, 0\rangle$, seen by acting the ladder operator $S_- = S_{1-} + S_{2-}$ onto the first state.

(b)

The total spin of uu must be 1, so the proton wavefunction can be seen as a composite of two subsystems of spin 1 and spin $\frac{1}{2}$ respectively. Proton has total spin $\frac{1}{2}$, using the Clebsch-Gordan table we find for $J, s_z = \frac{1}{2}, \frac{1}{2}$,

$$\begin{aligned}
\psi_p\left(s_z = +\frac{1}{2}\right) &= \sqrt{\frac{2}{3}}\,|1,1\rangle_{uu}\left|\frac{1}{2}, -\frac{1}{2}\right\rangle_d - \sqrt{\frac{1}{3}}\,|1,0\rangle_{uu}\left|\frac{1}{2}, \frac{1}{2}\right\rangle_d \\
&= \sqrt{\frac{2}{3}} u\uparrow u\uparrow d\downarrow - \frac{1}{\sqrt{6}}(u\uparrow u\downarrow + u\downarrow u\uparrow) d\uparrow \\
&= \frac{1}{\sqrt{6}}(2u\uparrow u\uparrow d\downarrow - u\uparrow u\downarrow d\uparrow + u\downarrow u\uparrow d\uparrow)
\end{aligned}$$

Similarly, the neutron wavefunction is symmetric under exchanging the two down quarks and its total spin is $\frac{1}{2}$,

$$\psi_n\left(s_z = +\frac{1}{2}\right) = \frac{1}{\sqrt{6}}(2u\downarrow d\uparrow d\uparrow - u\uparrow d\uparrow d\downarrow + u\uparrow d\downarrow d\uparrow)$$

(c)

Quarks are point-like spin-$\frac{1}{2}$ elementary particles, their magnetic moments are

$$\mu_q = \frac{q_q \hat{S}_z}{m_q} = \pm\frac{q_q \hbar}{2m_q}$$

The baryon magnetic moment operator is $\mu_b = \sum \mu_q$ over all quarks. For mixed states this is

$$\mu_b = \langle\psi|\,\mu_b\,|\psi\rangle$$

Assuming both the up and down quarks have the same mass m_q,

$$\mu_p = \frac{4}{6}\left(\frac{2}{3}+\frac{2}{3}+\frac{1}{3}\right)\frac{e\hbar}{2m_q} + \frac{1}{6}\left(\frac{2}{3}-\frac{2}{3}-\frac{1}{3}\right)\frac{e\hbar}{2m_q} + \frac{1}{6}\left(-\frac{2}{3}+\frac{2}{3}-\frac{1}{3}\right)\frac{e\hbar}{2m_q} = \frac{e\hbar}{2m_q}$$
$$\mu_n = \frac{4}{6}\left(-\frac{2}{3}-\frac{1}{3}-\frac{1}{3}\right)\frac{e\hbar}{2m_q} + \frac{1}{6}\left(\frac{2}{3}-\frac{1}{3}+\frac{1}{3}\right)\frac{e\hbar}{2m_q} + \frac{1}{6}\left(\frac{2}{3}+\frac{1}{3}-\frac{1}{3}\right)\frac{e\hbar}{2m_q} = -\frac{2}{3}\frac{e\hbar}{2m_q}$$

thence

$$\frac{\mu_p}{\mu_n} = -\frac{3}{2}.$$

(d)

The predicted ratio of proton and neutron magnetic moments above of -1.5 is not too far away from expected ratio $-\frac{2.79\mu_N}{1.91\mu_N}$. To fit the observed values, fix the quark masses

$$\mu_p = \frac{4}{6}\left(\frac{2m_d}{3m_u}+\frac{2m_d}{3m_u}+\frac{1}{3}\right)\frac{e\hbar}{2m_d} + \frac{1}{6}\left(\frac{2m_d}{3m_u}-\frac{2m_d}{3m_u}-\frac{1}{3}\right)\frac{e\hbar}{2m_d} + \frac{1}{6}\left(-\frac{2m_d}{3m_u}+\frac{2m_d}{3m_u}-\frac{1}{3}\right)\frac{e\hbar}{2m_d}$$
$$\mu_p = \frac{8}{9}\frac{e\hbar}{2m_u} + \frac{1}{9}\frac{e\hbar}{2m_d} = \left(\frac{8}{9}\frac{m_p}{m_u}+\frac{1}{9}\frac{m_p}{m_d}\right)\mu_N = 2.79\mu_N$$
$$\mu_n = \frac{4}{6}\left(-\frac{2m_d}{3m_u}-\frac{1}{3}-\frac{1}{3}\right)\frac{e\hbar}{2m_d} + \frac{1}{6}\left(\frac{2m_d}{3m_u}-\frac{1}{3}+\frac{1}{3}\right)\frac{e\hbar}{2m_d} + \frac{1}{6}\left(\frac{2m_d}{3m_u}+\frac{1}{3}-\frac{1}{3}\right)\frac{e\hbar}{2m_d}$$
$$\mu_n = -\frac{2}{9}\frac{e\hbar}{2m_u} - \frac{4}{9}\frac{e\hbar}{2m_d} = -\left(\frac{2}{9}\frac{m_p}{m_u}+\frac{4}{9}\frac{m_p}{m_d}\right)\mu_N = -1.91\mu_N$$
$$\implies m_u = 0.36m_p \qquad m_d = 0.34m_p$$

(e)

The wavefunctions of Σ^+ and Σ^- in their $s_z = +\frac{1}{2}$ states are completely analogous

$$\psi_{\Sigma^+} = \frac{1}{\sqrt{6}}(2u\uparrow u\uparrow s\downarrow - u\uparrow u\downarrow s\uparrow + u\downarrow u\uparrow s\uparrow)$$
$$\psi_{\Sigma^-} = \frac{1}{\sqrt{6}}(2d\uparrow d\uparrow s\downarrow - d\uparrow d\downarrow s\uparrow + d\downarrow d\uparrow s\uparrow)$$

so

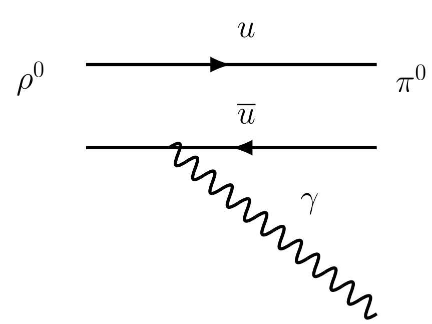
$$\mu_{\Sigma^+} = \frac{1}{6}\left(4\frac{4}{3}\frac{e\hbar}{2m_u} + 2\frac{1}{3}\frac{e\hbar}{2m_s}\right)$$

$$\mu_{\Sigma^+} = \frac{1}{6}\left(-4\frac{2}{3}\frac{e\hbar}{2m_d} + 2\frac{1}{3}\frac{e\hbar}{2m_s}\right)$$
$$\mu_{\Sigma^+} - \mu_{\Sigma^-} = \frac{8}{9}\frac{e\hbar}{2m_u} + \frac{4}{9}\frac{e\hbar}{2m_d}$$
$$\mu_p - \mu_n = \frac{10}{9}\frac{e\hbar}{2m_u} + \frac{5}{9}\frac{e\hbar}{2m_d}$$
$$\mu_{\Sigma^+} - \mu_{\Sigma^-} = \frac{4}{5}\left(\mu_p - \mu_n\right)$$

Problem 18

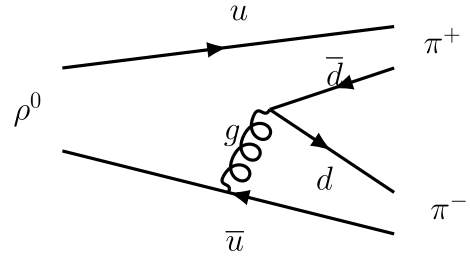
ρ^0 meson has $J^p = 1^-$ and $m_\rho = 775$ MeV

(a)



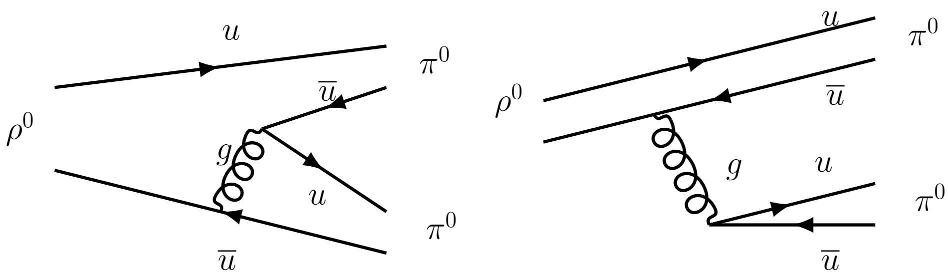
γ has $J^p = 1^-$ and pions have $J^p = 0^-$. It is possible to conserve parity and angular momentum using $l = 1$. The pion has mass 134 MeV so it is also possible to conserve energy-momentum. The process is <u>allowed</u>.

(b)



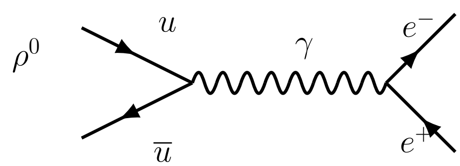
Charged pions have $J^p = 0^-$ and mass 139 MeV. Putting the product in state $l = 1$ allows us to conserve angular momentum, parity and energy-momentum. The process is <u>allowed</u>.

(c)



This process has the same conditions as the process in part (b) but π_0s are spin-0 identical bosons they cannot be put in spatially antisymmetric $l = 1$ state and the process is forbidden.

(d)



Electron has only 0.510 MeV rest mass, invariant mass can be conserved. Charge is also trivially conserved. To conserve initial ρ^0 spin angular momentum 1, we need $L - S \leq 1 \leq L + S$. The parity of the lepton pair is equal to the parity of ρ^0

$$(+1)(-1)(-1)^l = (-1)^{l+1} = -1$$

So $L = 0$ or 2 and $S = 1$ can conserve angular momentum and parity. The decay is allowed.

The three allowed modes, ranked by expected rates from high to low are

mode	coupling strength
(b)	$\sqrt{\alpha_s}\sqrt{\alpha_s}$
(a)	$u\overline{u}$: $\frac{2}{3}\sqrt{\alpha}$ or $d\overline{d}$: $-\frac{1}{3}\sqrt{\alpha}$
(d)	$\frac{2}{3}\sqrt{\alpha}\sqrt{\alpha}$ or $-\frac{1}{3}\sqrt{\alpha}\sqrt{\alpha}$

If we consider charges of particles as eigenvalues of the charge operator Q, the electromagnetic coupling strength from QED is

$$\text{coupling strength} = \langle \text{final} | \, Q\sqrt{\alpha} \, | \text{initial} \rangle \tag{1}$$

Consider the decay of mesons ρ^0 and ω^0. They have roughly the same mass, so propagator effects can be omitted. They also both have $J^p = 1^-$.

Set $|\text{initial}\rangle = \frac{1}{\sqrt{2}}(u\overline{u} \mp d\overline{d})$ for ρ_0 and ω_0, and $|\text{final}\rangle = \frac{1}{\sqrt{2}}(u\overline{u} - d\overline{d})$ for π_0, and let the matrix elements for decay to $\pi^0\gamma$ via *(a)* are respectively

$$\rho^0 \to \pi^0\gamma: \qquad M = \frac{1}{2}\left(\frac{2}{3}\sqrt{\alpha} - \frac{1}{3}\sqrt{\alpha}\right) = \frac{1}{6}\sqrt{\alpha}$$
$$\omega^0 \to \pi^0\gamma: \qquad M = \frac{1}{2}\left(\frac{2}{3}\sqrt{\alpha} + \frac{1}{3}\sqrt{\alpha}\right) = \frac{3}{6}\sqrt{\alpha}$$

The matrix elements differ by a factor of 3 so the partial widths differ by a factor of ~ 10.

To calculate the matrix elements for decay to e^+e^- via *(b)*, we cannot trivially use eq. (1) because $|\text{initial}\rangle$ and $|\text{final}\rangle$ are entangled. Instead, try to sum up the two possible diagrams

$$\rho^0 \to e^+e^-: \qquad M = \frac{1}{2}\underbrace{\frac{2}{3}\sqrt{\alpha}\sqrt{\alpha}}_{u\overline{u}} - \frac{1}{2}\underbrace{\left(-\frac{1}{3}\sqrt{\alpha}\sqrt{\alpha}\right)}_{d\overline{d}} = \frac{3}{6}\sqrt{\alpha}\sqrt{\alpha}$$
$$\omega^0 \to e^+e^-: \qquad M = \frac{1}{2}\underbrace{\frac{2}{3}\sqrt{\alpha}\sqrt{\alpha}}_{u\overline{u}} + \frac{1}{2}\underbrace{\left(-\frac{1}{3}\sqrt{\alpha}\sqrt{\alpha}\right)}_{d\overline{d}} = \frac{1}{6}\sqrt{\alpha}\sqrt{\alpha}$$

so the corresponding partial widths differ by a factor of $3^2 \sim 10$.

However, this is nonphysical. $u\overline{u}$ and $d\overline{d}$ are orthogonal states whose relative phases are irrelevant.

Problem 19

(a)

Using the Breit-wigner formula for J/ψ resonance in elastic scattering of e^+e^-, approximate λ as a constant value near the resonance

$$\sigma(E) = \frac{\lambda^2}{4\pi}\frac{2J+1}{(2s_1+1)(2s_2+1)}\frac{\Gamma_{e^+e^-}\Gamma_{e^+e^-}}{(E-E_0)^2+\Gamma^2/4}$$
$$\sigma' = \int \frac{\lambda^2}{4\pi}\frac{2J+1}{(2s_1+1)(2s_2+1)}\frac{\Gamma_{e^+e^-}\Gamma_{e^+e^-}}{(E-E_0)^2+\Gamma^2/4}\,\mathrm{d}E$$
$$\sigma' = \frac{\lambda^2}{4}\frac{2J+1}{(2s_1+1)(2s_2+1)}\frac{2}{\Gamma}\Gamma^2_{e^+e^-}$$
$$\sigma' = \frac{\lambda^2}{2}\frac{2\times 1+1}{(2\times\frac{1}{2}+1)(2\times\frac{1}{2}+1)}B^2\Gamma$$
$$\sigma' = \frac{3}{8}\lambda^2 B^2\Gamma$$

(b)

Assuming that at each scan point of mean energy E the beam energy E' is described by probability distribution $f(E'-E)$, the expectation $\mathfrak{E}$ measured area under the resonance peak will be

$$\begin{aligned}\mathfrak{E}\left[\int \sigma_{meas}(E)\,\mathrm{d}E\right] &= \int f(E'-E)\int \sigma(E)\,\mathrm{d}E\,\mathrm{d}E' \\ &= \int\int f(E'-E)\,\mathrm{d}E'\,\sigma(E)\,\mathrm{d}E \\ &= \int \sigma(E)\,\mathrm{d}E\end{aligned}$$

which is the true area under the peak.

(c)

The differential cross-section is proportional to $1+\cos^2\theta$. The relative number of particles in a range of θ is thus

$$\begin{aligned}\int \mathrm{d}\Omega\left(1+\cos^2\theta\right) &= 2\pi\int \mathrm{d}\theta\sin\theta\left(1+\cos^2\theta\right) \\ \int \mathrm{d}\Omega\left(1+\cos^2\theta\right) &= -2\pi\left[\cos\theta+\frac{\cos^3\theta}{3}\right]_{\text{range}} \\ \int_{|\cos\theta|<0.6} \mathrm{d}\Omega\left(1+\cos^2\theta\right) &= 2\pi\frac{504}{3\times 5^3} = 2\pi\frac{504}{375} = 2\pi\frac{504}{1000\times 3/8} \\ \int_{\text{sphere}} \mathrm{d}\Omega\left(1+\cos^2\theta\right) &= 2\pi\frac{8}{3} \\ \text{acceptance fraction} &= 50.4\%\end{aligned}$$

(d)

Using the area under (c) to obtain σ' and compare with the sum over (a,b,c) to get B. We might get

$$\begin{aligned}\Gamma &= \frac{8\sigma'}{3\lambda^2 B^2} \\ \Gamma &= \frac{8E_0^2\sigma'}{3(2\pi\hbar c)^2 B^2} \\ \Gamma &= 1.97\times 10^2\ \text{keV} \\ \Gamma_{ee} &= 1.28\times 10^1\ \text{keV}\end{aligned}$$

Neither coincide with the answers given.

(e)

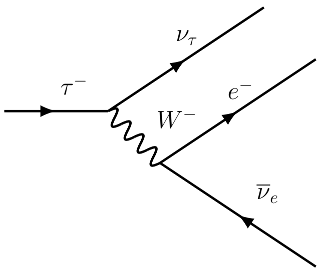
The leptonic widths Γ_{ee} of ϕ and J/ψ are similar, because both particles decay via the same mechanism of annihilating their quarks and creating electron-positron pair.

The total width Γ of ϕ is much greater than that of J/ψ because the later cannot decay to lighter states without complicated unconnected annihilation of charm quarks and creation of lighter quarks, which, according to Zweig rule, leads to suppressed decay amplitude; whilst the former can decay to lighter quarks via emitting an imaginary gluon and pair producing some quarks, with much fewer number of vertices.

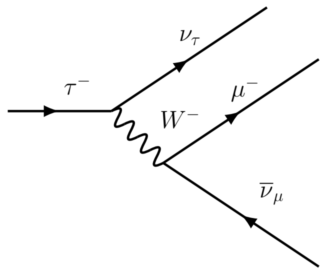
Topic 7 Weak interactions

Problem 20 Tau decay

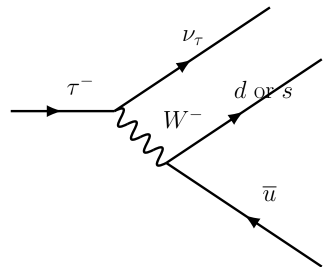
Factor away the matrix element associated with the $\tau\nu_\tau W^-$ vertex which is common to all three processes, In low energy limits $p^2 \ll m_W^2$, the expected ratios of the three partial widths



(a) Matrix element $\propto \frac{g_W}{p^2 - m_W^2}$



(b) Matrix element $\propto \frac{g_W}{p^2 - m_W^2}$



(c) Matrix element for $\overline{u}d \propto \frac{g_W \cos\theta_C}{p^2 - m_W^2}$ and for $\overline{u}s \propto \frac{g_W \sin\theta_C}{p^2 - m_W^2}$, *for each colour*

are

$$\left(-\frac{g_W}{m_W^2}\right)^2 : \left(-\frac{g_W}{m_W^2}\right)^2 : 3\left(-\frac{g_W \cos\theta_C}{m_W^2}\right)^2 + 3\left(-\frac{g_W \sin\theta_C}{m_W^2}\right)^2 = 1 : 1 : 3$$

The actual ratios are

$$1.02 : 1 : 3.5$$

The 1.02 might be due to $p_e > p_\mu$ slightly, because $m_e \ll m_\mu$. 3.5 might be attributable to higher order strong interactions between quarks.

Assuming both the partial widths of μ^- decay and τ^- decay to $\nu_{\mu/\tau} e^- \overline{\nu}_e$ obey Sargent's rule

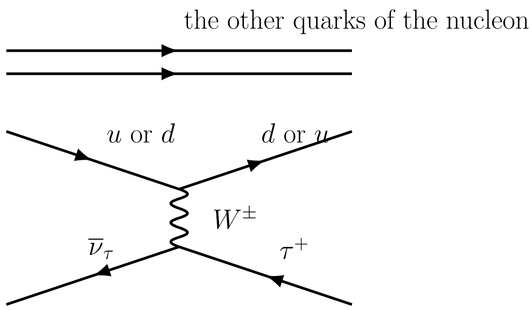
$$\Gamma_{->e} = \frac{G_F^2 E_0^5}{60\pi^3} \implies \frac{\Gamma_{\tau \to e}}{\Gamma_{\mu \to e}} = \left(\frac{m_\tau}{m_\mu}\right)^5$$

$$\tau_\tau = \frac{1}{\Gamma_{\tau \to e}/B(\tau \to e)} = B(\tau \to e)\tau_{\mu \to e}\left(\frac{m_\mu}{m_\tau}\right)^5 = 3 \times 10^{-13}\ \mathrm{s}$$

where we have adopted $\tau_{\mu \to e} = \tau_\mu$ because muon only has enough mass to decay to electron, not to muon itself plus two neutrinos or to the lightest meson.

Problem 21 Threshold energy

(a)



(b)

Conserving invariant mass, if the products are at rest in ZMF, and approximate neutrinos as massless

$$\left(E_{\overline{\nu}_\tau} + m_p\right)^2 - p_{\overline{\nu}_\tau}^2 = (m_n + m_{\tau^+})^2$$
$$m_{\overline{\nu}_\tau}^2 + 2E_{\overline{\nu}_\tau} m_p + m_p^2 = m_n^2 + 2m_n m_{\tau^+} + m_{\tau^+}^2$$
$$2E_{\overline{\nu}_\tau} m_n = 2m_n m_{\tau^+} + m_{\tau^+}^2 + m_n^2 - m_p^2$$
$$E_{\overline{\nu}_\tau} = m_{\tau^+}\left(1 + \frac{m_{\tau^+}}{2m_n}\right) + \frac{m_n^2 - m_p^2}{2m_n} = 3.46 \times 10^3 \text{ MeV}$$

(c)

In lab frame, the momentum of τ^+ is

$$p_{\tau^+} = \frac{m_{\tau^+}}{m_{\tau^+} + m_n} E_{\overline{\nu}_\tau} = 2.26 \times 10^3 \text{ MeV}$$

from which we get its energy

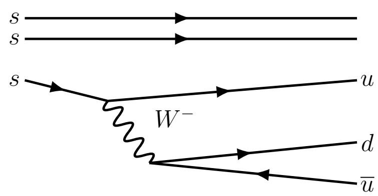
$$E_{\tau^+} = \sqrt{p_{\tau^+}^2 + m_{\tau^+}^2} = 2.88 \times 10^3 \text{ MeV}$$

(d)

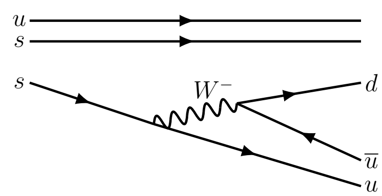
$$\beta_{\tau^+} = \frac{p_{\tau^+}}{E_{\tau^+}}; \qquad \gamma_{\tau^+} = \frac{E_{\tau^+}}{m_{\tau^+}}$$
$$l_{\tau^+} = \beta_{\tau^+}\gamma_{\tau^+}\tau_{\tau^+} = \frac{p_{\tau^+}}{m_{\tau^+}}\tau_{\tau^+} = 1.11 \times 10^{-4} \text{ m}$$

Problem 22 Omega decay

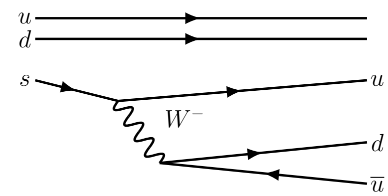
Weak decays of Ω^-



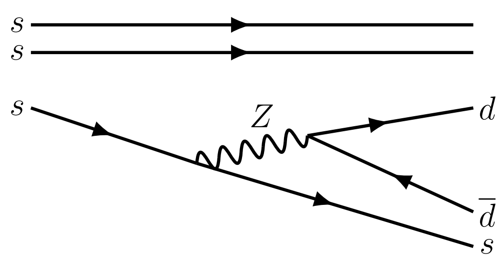
$\Omega^-(sss) \to \Xi^0(uss)\pi^-(\overline{u}d)$



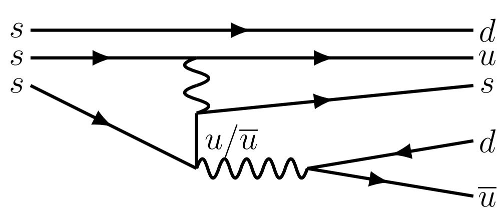
$\Xi^0(uss) \to \Lambda^0(usd)\pi^0(\overline{u}u)$



$\Lambda^0(usd) \to p(uud)\pi^-(\overline{u}d)$



$\Omega^-(sss) \to \Xi^-(ssd)\overline{K}_0(s\overline{d})$. The process is forbidden because the mass of the Omega baryon 1672 MeV is insufficient for the combined rest masses of Kaon 498 MeV and Xion 1321 MeV.

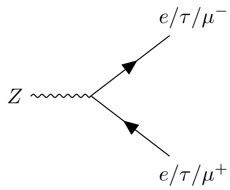


$\Omega^-(sss) \to \Lambda^0(usd)\pi^-(\bar{u}d)$. All the bosons are W^-. This process is is strongly suppressed because it consists of 4 vertices compared to the above two-vertex diagrams.

Topic 8 Electroweak unification

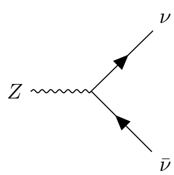
Problem 23

(a)



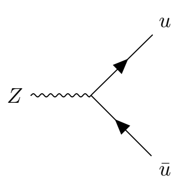
$$\Gamma_{l\bar{l}} \propto g_Z^2 \left[\left(0 - (-1)\sin^2\theta_W\right)^2 + \left(-\frac{1}{2} - (-1)\sin^2\theta_W\right)^2 \right]$$

$$= g_Z^2 \left(2\sin^4\theta_W + \frac{1}{4} - \sin^2\theta_W \right)$$



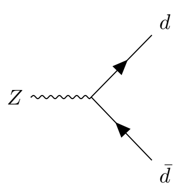
$$\Gamma_{\nu\bar{\nu}} \propto g_Z^2 \left[\left(0 - 0\sin^2\theta_W\right)^2 + \left(\frac{1}{2} - 0\sin^2\theta_W\right)^2 \right]$$

$$= g_Z^2 \frac{1}{4}$$



$$\Gamma_{u\bar{u}} \propto g_Z^2 \left[\left(0 - \frac{2}{3}\sin^2\theta_W\right)^2 + \left(\frac{1}{2} - \frac{2}{3}\sin^2\theta_W\right)^2 \right]$$

$$= g_Z^2 \left(\frac{8}{9}\sin^4\theta_W + \frac{1}{4} - \frac{2}{3}\sin^2\theta_W \right)$$



$$\Gamma_{d\bar{d}} \propto g_Z^2 \left[\left(0 + \frac{1}{3}\sin^2\theta_W\right)^2 + \left(-\frac{1}{2} + \frac{1}{3}\sin^2\theta_W\right)^2 \right]$$

$$= g_Z^2 \left(\frac{2}{9}\sin^4\theta_W + \frac{1}{4} - \frac{1}{3}\sin^2\theta_W \right)$$

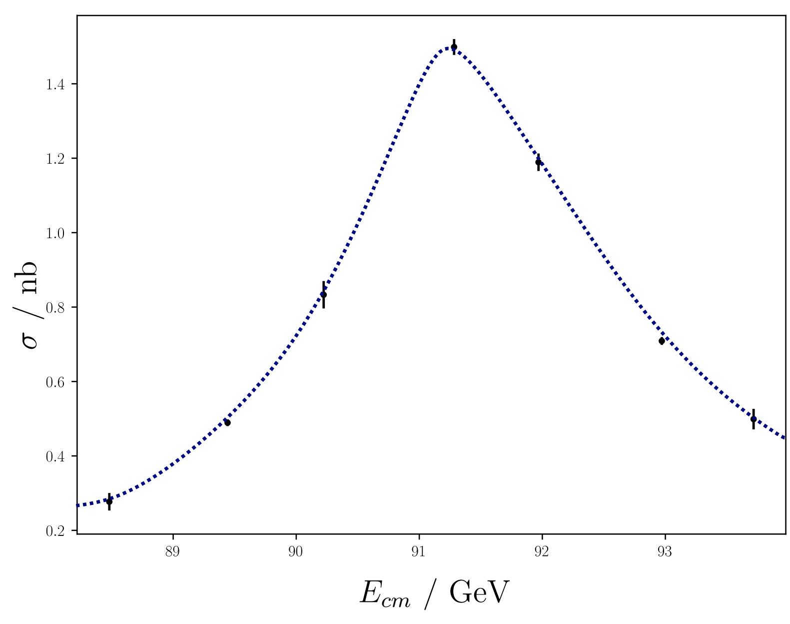
The real Z boson can decay to all three generations of leptons and their neutrinos, as well as u, c, d, s, b quarks (top quark is too heavy).

$$\Gamma_Z = \left(2\sin^4\theta_W + \frac{1}{4} - \sin^2\theta_W\right)\times 3 + \frac{1}{4}\times 3$$
$$+\left[\left(\frac{8}{9}\sin^4\theta_W + \frac{1}{4} - \frac{2}{3}\sin^2\theta_W\right)\times 2 + \left(\frac{2}{9}\sin^4\theta_W + \frac{1}{4} - \frac{1}{3}\sin^2\theta_W\right)\times 3\right]\times 3$$

The branching fraction for Z decay to

$$\tau^+\tau^- : \quad B = \frac{2\sin^4\theta_W + \frac{1}{4} - \sin^2\theta_W}{\Gamma_Z/g_Z^2} = 0.034$$
$$\text{(for each one generation of) } \nu\bar{\nu} : \quad B = \frac{1/4}{\Gamma_Z/g_Z^2} = 0.068$$
$$\text{(for each one generation of) } u\bar{u} : \quad B = \frac{\left(\frac{8}{9}\sin^4\theta_W + \frac{1}{4} - \frac{2}{3}\sin^2\theta_W\right)\times 3}{\Gamma_Z/g_Z^2} = 0.118$$
$$\text{(for each one generation of) } d\bar{d} : \quad B = \frac{\left(\frac{2}{9}\sin^4\theta_W + \frac{1}{4} - \frac{1}{3}\sin^2\theta_W\right)\times 3}{\Gamma_Z/g_Z^2} = 0.152$$
$$\text{All hadrons} : \quad B = B_{u\bar{u}}\times 2 + B_{d\bar{d}}\times 3 = 0.692$$

(b)



$$\sigma^0 = 1.5\times 10^{-37}\ \text{m}^2$$
$$M_Z = 91\ \text{GeV}$$
$$\Gamma_Z = 2.5\ \text{GeV}$$

$$\sigma^0 = \frac{12\pi}{M_Z^2}\frac{\Gamma_\tau \Gamma_e}{\Gamma_Z^2} \implies \Gamma_\tau = \sqrt{\frac{\sigma^0 M_Z^2 \Gamma_Z^2}{12\pi}} = 73 \text{ MeV}$$
$$B_\tau = 0.029$$

Considering our rough interpolation, this experimental value of branching factor is well consistent with electroweak theory.

The electrons/positions radiate away their energy via synchrotron radiation which is energy dependent. As a result, the real cm energies are shifted with respect to their apparent value, causing asymmetry of the resonance curve.

(c)

Assuming lepton universality, and the Z has enough mass to decay to all three generations of leptons,

$$\frac{\sigma_{e\to Z\to \text{hadrons}}}{\sigma_{e\to Z\to \mu}} = \frac{\Gamma_{\text{hadrons}}}{\Gamma_\mu} = 20.7$$
$$\Gamma_Z = \Gamma_\mu \times 3 + \Gamma_\nu \times 3 + \frac{\Gamma_{\text{hadrons}}}{\Gamma_\mu}\Gamma_\mu = 2.47 \times 10^3 \text{ eV}$$

Problem 24

Error of the result is dominated by that from the total width.

$$\Gamma(W^- \to e^-\bar{\nu}_e) = \frac{G_F}{\sqrt{2}}\frac{M_W^3}{6\pi} = 2.34 \times 10^{-1} \text{ GeV}$$

Assuming the W^- couples equally to quarks and leptons, and that it is only massive enough to decay to $d'\bar{u}$ and $s'\bar{c}$,

$$\Gamma(W^- \to d'\bar{u}) = 3 \times \Gamma(W^- \to e^-\bar{\nu}_e) + 3 \times \Gamma(W^- \to e^-\bar{\nu}_e) = 6\Gamma(W^- \to e^-\bar{\nu}_e)$$

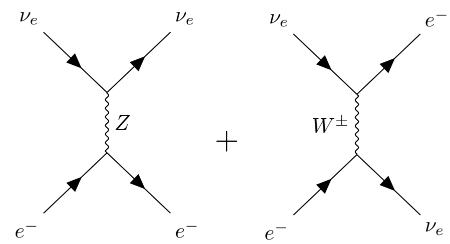
Considering three colour multiplicity for each quark. The total lepton decay width is therefore

$$\frac{\Gamma(W^- \to l\bar{\nu}_l)}{\Gamma(W^- \to e^-\bar{\nu}_e)} = 8.92 - 6 = 2.92 \pm 0.2$$

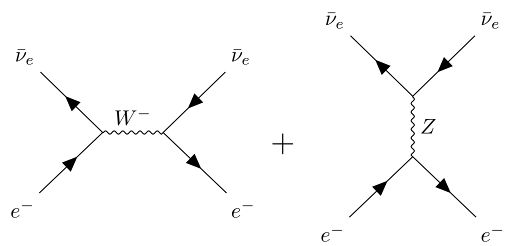
The number of lepton generations is estimated to be 3.

Problem 25

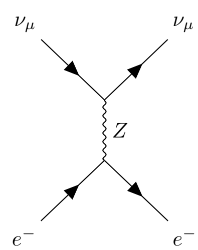
(a)



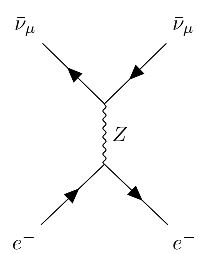
(b)



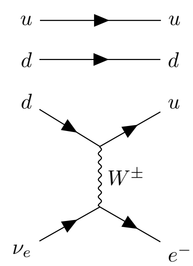
(c)



(d)



(e)

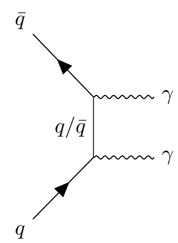


Problem 26

(a)

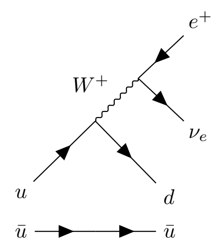
Decays of π^0

to $\gamma\gamma$:



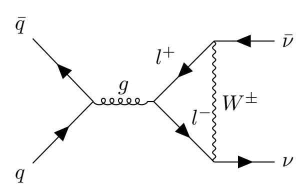
$$\Gamma \propto \left(e\frac{2}{3}\right)^2 + \left(e\frac{1}{3}\right)^2 + \left(e\frac{2}{3}\right)^{2^2}$$

to $\pi^-\nu_e e^+$:



forbidden because $\pi^\pm$ are heavier than π^0

to $\nu\bar{\nu}$:

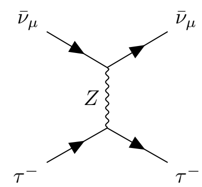


The oppositely moving left-handed neutrino and right-handed antineutrino give nonzero total helicity in CM frame $\implies$ forbidden

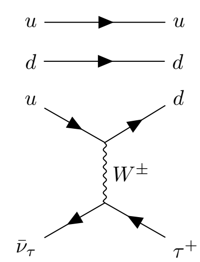
(b)

e^+e^- pair cannot decay to $\tau^+\tau^-$ which is a heavier generation.

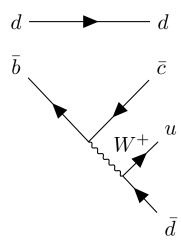
The decay $\bar{\nu}_\mu + \tau^- \to \bar{\nu}_\mu + \tau^-$ has only one possible (lowest order) Feynman diagram



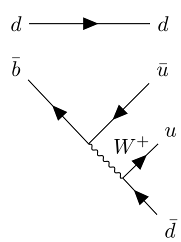
The decay $p + \nu_\tau \to n + \tau^+$ does not conserve lepton number, so it does not happen. The close $p + \bar{\nu}_\tau \to n + \tau^+$ is allowed via



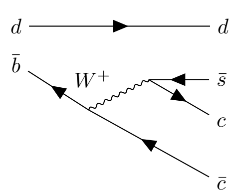
(c)



$$\Gamma \propto g_W^4 (V_{bc} V_{ud})^2 = g_W^4 (\sin^2\theta_C \cos\theta_C)^2$$



$$\Gamma \propto g_W^4 (V_{bu} V_{ud})^2 = g_W^4 (\sin^3\theta_C \cos\theta_C)^2$$

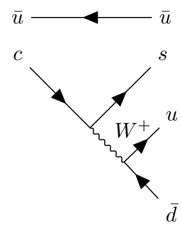


$$\Gamma \propto g_W^4 (V_{bc} V_{cs})^2 = g_W^4 (\sin^2\theta_C \cos\theta_C)^2$$

The interaction strengths of B^0 decay to $J/\psi K^0$ and to $D\pi^+$ are the same, while the decay to $\pi^+\pi^-$ is Cabibbo suppressed significantly.

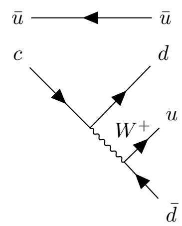
(d)

D^0 decay to $K^-\pi^+$



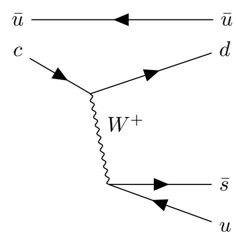
$\Gamma \propto g_W^4 (V_{sc}V_{ud})^2 = g_W^4 (\cos\theta_C \cos\theta_C)^2$

D^0 decay to $\pi^+\pi^-$



$\Gamma \propto g_W^4 (V_{cd}V_{ud})^2 = g_W^4 (-\sin\theta_C \cos\theta_C)^2$

D^0 decay to $K^+\pi^-$



$\Gamma \propto g_W^4 (V_{cd}V_{us})^2 = g_W^4 (-\sin\theta_C \sin\theta_C)^2$

The reactions are more Cabibbo suppressed down the table.

Problem 27

(a)

Let $\boldsymbol{\nu}_2$ and $\boldsymbol{\nu}_2$ be normalised to 1.

$$\begin{aligned}
\boldsymbol{\nu}_\mu &= \boldsymbol{\nu}_2 \cos\theta + \boldsymbol{\nu}_3 \sin\theta \\
\boldsymbol{\psi}(L=0) &= \boldsymbol{\nu}_\mu \\
\boldsymbol{\psi}(t) &= \exp\left(\frac{iHt}{\hbar}\right)\boldsymbol{\psi}(L=0) \\
P_\mu(L) &= \left|\boldsymbol{\nu}_\mu \cdot \left(\exp\left(\frac{iE_2L}{\hbar c}\right)\boldsymbol{\nu}_2\cos\theta + \exp\left(\frac{iE_3L}{\hbar c}\right)\boldsymbol{\nu}_3\sin\theta\right)\right|^2 \\
P_\mu(L) &= \left|\exp\left(\frac{iE_2L}{\hbar c}\right)\cos^2\theta + \exp\left(\frac{iE_3L}{\hbar c}\right)\sin^2\theta\right|^2 \\
P_\mu(L) &= \cos^4\theta + \sin^4\theta + 2\cos^2\theta\sin^2\theta\cos\left(\frac{(E_2-E_3)L}{\hbar c}\right)
\end{aligned}$$

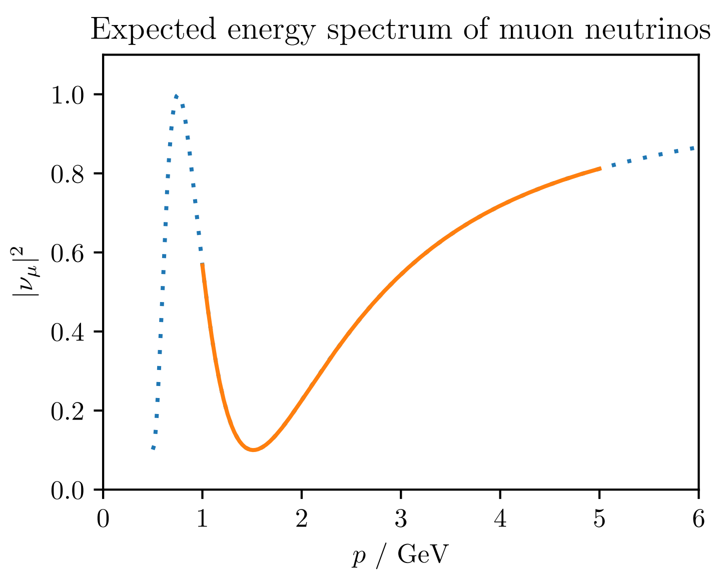
$$P_\mu(L) = \cos^4\theta + \sin^4\theta + 2\cos^2\theta\sin^2\theta - 4\cos^2\theta\sin^2\theta\sin^2\left(\frac{(E_2-E_3)L}{2\hbar c}\right)$$
$$P_\mu(L) = 1 - \sin^2(2\theta)\sin^2\left(\frac{(E_2-E_3)L}{2\hbar c}\right)$$
$$N_\mu(L) = N_\mu(L=0)\left[1 - \sin^2(2\theta)\sin^2\left(\frac{(E_2-E_3)L}{2\hbar c}\right)\right]$$

(b)

If $m_2, m_3 \ll p$,

$$E_2 - E_3 = \sqrt{p^2+m_2^2} - \sqrt{p^2+m_3^2}$$
$$E_2 - E_3 = \frac{m_2^2 - m_3^2}{2p}$$
$$N_\mu(L) \approx N_\mu(L=0)\left[1 - \sin^2(2\theta)\sin^2\left(\frac{1}{4\hbar c}\frac{(m_2^2-m_3^2)L}{p}\right)\right]$$

(c)



(d)

The threshold energy for creation of τ, from Qu.21, is 3.46 GeV. Above this energy, the probability for oscillating into τ is $\approx (0.2, 0.4)$. Some τ leptons could be detected at MINOS.

Topic 9

Problem 28

(a)

Bookwork

(b)

(c)

(d)

(e)

Scale up to neutron star.

There's no bound state of just neutron. There's only one bound state of proton (Hydrogen). In the neutron star Gravitational effects come in allow bound state of neutrons.

The binding energy has to be positive.

Problem 29

$$g_\epsilon = BA\epsilon^{1/2}$$
$$N = \int_0^{\epsilon_F} \mathrm{d}\epsilon\, g_\epsilon = \frac{2}{3}BA\epsilon_F^{3/2}$$
$$\epsilon_F = \left(\frac{3N}{2BA}\right)^{2/3}$$

For $N = Z = \frac{1}{2}A$,

$$\epsilon_F = \left(\frac{3}{4B}\right)^{2/3}$$
$$\epsilon_F = \left(\frac{9\pi}{16\sqrt{2}}\right)^{2/3}\frac{\hbar^2}{mR_0^2}$$
$$\epsilon_F = 33.4 \text{ MeV}$$

The total kinetic energy is

$$\begin{aligned} E &= \int_0^{\epsilon_F(N)} g_\epsilon \epsilon \,\mathrm{d}\epsilon + \int_0^{\epsilon_F(Z)} g_\epsilon \epsilon\,\mathrm{d}\epsilon \\ &= \int_0^{\epsilon_F(N)} BA\epsilon^{3/2}\,\mathrm{d}\epsilon + \int_0^{\epsilon_F(Z)} BA\epsilon^{3/2}\,\mathrm{d}\epsilon \end{aligned}$$

$$= \frac{2}{5} BA \left[\epsilon_F^{5/2}(N) + \epsilon_F^{5/2}(Z) \right]$$
$$= \frac{2}{5} BA \left(\frac{3}{2BA} \right)^{5/3} \left[N^{5/2} + Z^{5/2} \right]$$
$$= \frac{3}{5} \left(\frac{3}{2BA} \right)^{2/3} \left[N^{5/2} + Z^{5/2} \right]$$

Problem 30

Nuclear potential rounded corners top-hat Saxon potential

Nuclear shell potential

Intensity $\propto F^2 = 0.7$

Problem 31

Coulomb term

Problem 32

Problem 33

Magic number

stuff dont agree

sodium is not spherically symmetric