Relativity Example Sheets

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Michaelmas 2020

Contents

Example Sheet 1

Example 1.1

Without loss of generality, we consider systems of reference in which y and z coordinates are perpendicular to the connecting line of events of interest in spacetime.

$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2$$

Transform rules:

$$c\Delta t' = \gamma(c\Delta t\ - \beta\Delta x) = c\Delta t\cosh\psi_u - \Delta x\sinh\psi_u$$
$$\Delta x' = \gamma(\Delta x\ - \beta c\Delta t) = \Delta x\cosh\psi_u - c\Delta t\sinh\psi_u$$

Proof by construction:

(a)

time-like: $\Delta s^2 > 0$

$$c^2\Delta t^2 - \Delta x^2 > 0$$
$$-1 < \frac{\Delta x}{c\Delta t} < 1$$

To find S' where $\Delta x = 0$, we simp;y require $\frac{\Delta x}{c\Delta t} = \tanh\psi_u$ which can always be found for real rapidity $-1 < \psi_u < 1$.

(b)

space-like: $\Delta s^2 < 0$

$$c^2\Delta t^2 - \Delta x^2 < 0$$
$$-1 < \frac{c\Delta t}{\Delta x} < 1$$

To find S' where $\Delta t = 0$, we simp;y require $\frac{c\Delta t}{\Delta x} = \tanh\psi_u$ which can always be found for real rapidity $-1 < \psi_u < 1$.

Example 1.2

(a)

In S, $\Delta t = t_B - t_A > 0$, $\Delta x = 0$.

In all frames S',

$$\Delta t' = \Delta t\cosh\psi_u$$
$$\Delta t' \geq \Delta t > 0$$
$$t'_B > t'_A$$

(b)

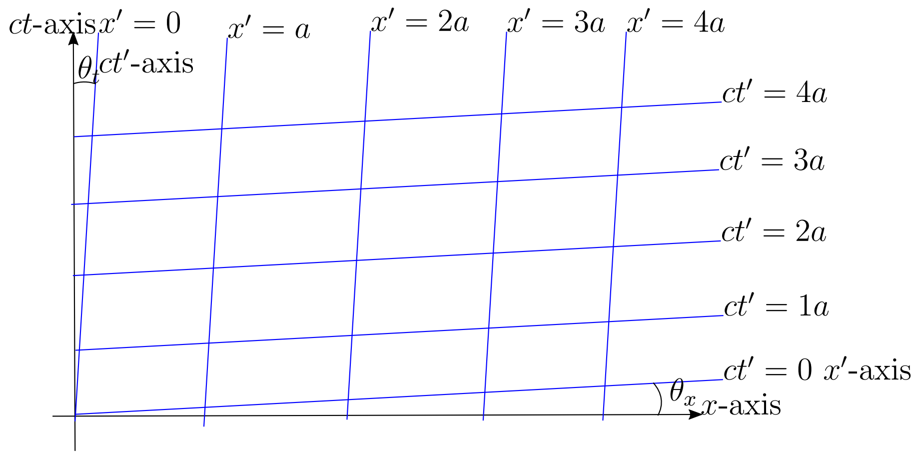
If event A causes event B, $\Delta t = t_B - t_A \geq \frac{\Delta r}{c} \geq 0$,

$$\Delta t = \Delta t \cosh \psi_u - \frac{\Delta r}{c} \sinh \psi_u \geq \Delta t(\cosh \psi_u - \sinh \psi_u) \geq 0$$

$$\begin{gathered}\Delta s^2 = c^2 \Delta t^2 - \Delta^2 \geq 0 \\ c^2 \Delta t'^2 - \Delta r'^2 \geq 0 \\ \Delta t \geq \frac{|\Delta r'|}{c} \quad \text{in all frames.} \qquad \blacksquare\end{gathered}$$

Example 1.3

(a)



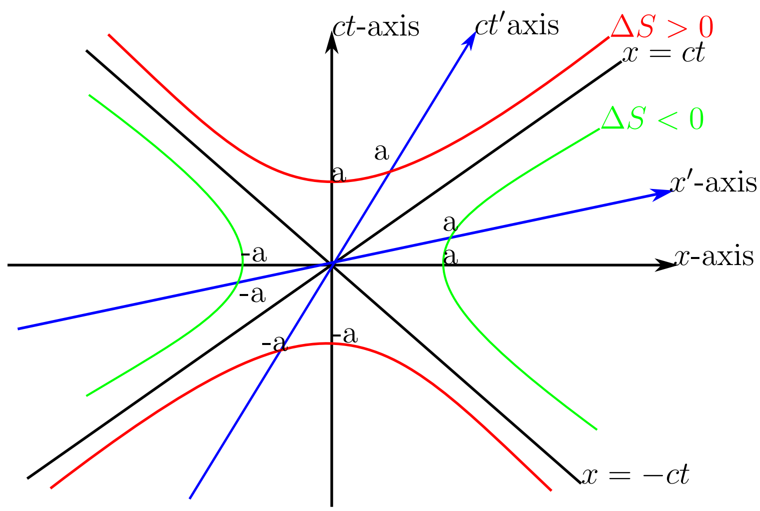
$$\begin{gathered}x' = x \cosh \psi_v - ct \sinh \psi_v \\ ct'\text{-axis:} \quad x \cosh \psi_v - ct \sinh \psi_v = 0 \\ \theta_t = \tan^{-1}\left(\frac{\sinh \psi_v}{\cosh \psi_v}\right) \\ \theta_t = \tan^{-1}\left(\frac{\beta\gamma}{\gamma}\right) = \tan^{-1}\left(\frac{v}{c}\right)\end{gathered}$$

Similarly

$$\begin{gathered}ct' = ct \cosh \psi_v - x \sinh \psi_v \\ x'\text{-axis:} \quad ct \cosh \psi_v - x \sinh \psi_v = 0\end{gathered}$$

$$\theta_x = \tan^{-1}\left(\frac{ct}{x}\right)$$
$$\theta_x = \tan^{-1}\left(\frac{\beta\gamma}{\gamma}\right) = \tan^{-1}\left(\frac{v}{c}\right)$$

(b)



$$\Delta s^2 = c^2t^2 - x^2$$

Since we are interested in constant Δs^2 curves,

$$\left(\frac{\partial \Delta s^2}{\partial x}\right)_{\Delta s^2} = 0 = 2ct\left(\frac{\partial ct}{\partial x}\right)_{\Delta s^2} - 2x$$

If the curve does intersect the ct-axis, at $x = 0$, we have

$$\left(\frac{\partial ct}{\partial x}\right)_{\Delta s^2} = 0 \implies \text{curve is parallel to } x\text{-axis}$$

Similarly, taking derivative with respect to ct, we get

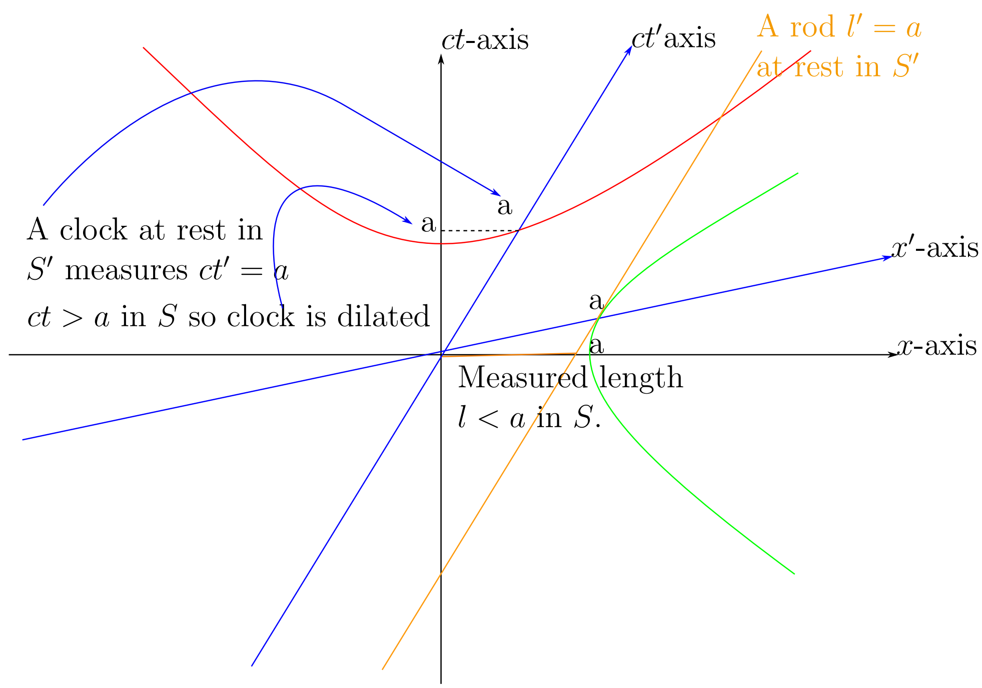
$$\left(\frac{\partial \Delta s^2}{\partial ct}\right)_{\Delta s^2} = 0 = 2x\left(\frac{\partial x}{\partial ct}\right)_{\Delta s^2} - 2ct$$

which means at $ct = 0$ (intersecting x-axis)

$$\left(\frac{\partial x}{\partial ct}\right)_{\Delta s^2} = 0 \implies \text{curve is parallel to } ct\text{-axis}$$

These curves intersect the coordinate axes of different S' frames at the same values of x' or t', as shown in the plot above. The new axes can then be calibrated linearly with respect to the test length $x' = \sqrt{-\Delta s^2}, ct' = \sqrt{\Delta s^2}$

(c)



Example 1.4

Dissolve the 3-vector coordinate $\mathbf{r} = (x, y, z)^T$ into components parallel and perpendicular to β

$$\vec{r} = \overbrace{\frac{\mathbf{r}\cdot\beta}{\beta^2}\vec{\beta}}^{\text{parallel}} + \overbrace{\vec{r} - \frac{\mathbf{r}\cdot\beta}{\beta^2}\vec{\beta}}^{\text{perpendicular}}$$

$$r_\parallel = \frac{(\beta_x x + \beta_y y + \beta_z z)}{\beta} \qquad \vec{r}_\perp = \begin{pmatrix} x - \frac{(\beta_x x + \beta_y y + \beta_z z)}{\beta^2}\beta_x \\ y - \frac{(\beta_x x + \beta_y y + \beta_z z)}{\beta^2}\beta_y \\ z - \frac{(\beta_x x + \beta_y y + \beta_z z)}{\beta^2}\beta_z \end{pmatrix}$$

Then the rules for the components can be applied respectively:

$$ct' = \gamma(ct - \beta r_\parallel)$$
$$\vec{r} = \gamma(r_\parallel - \beta ct)\frac{\vec{\beta}}{\beta} + \vec{r}_\perp$$

Reorganised into matrix equations

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & \gamma\frac{\beta_x^2}{\beta^2} & \gamma\frac{\beta_x\beta_y}{\beta^2} & \gamma\frac{\beta_x\beta_z}{\beta^2} \\ -\gamma\beta_y & \gamma\frac{\beta_y\beta_x}{\beta^2} & \gamma\frac{\beta_y^2}{\beta^2} & \gamma\frac{\beta_y\beta_z}{\beta^2} \\ -\gamma\beta_z & \gamma\frac{\beta_z\beta_x}{\beta^2} & \gamma\frac{\beta_z\beta_y}{\beta^2} & \gamma\frac{\beta_z^2}{\beta^2} \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 - \frac{\beta_x^2}{\beta^2} & -\frac{\beta_y\beta_x}{\beta^2} & -\frac{\beta_z\beta_x}{\beta^2} \\ 0 & -\frac{\beta_x\beta_y}{\beta^2} & 1 - \frac{\beta_y^2}{\beta^2} & -\frac{\beta_z\beta_y}{\beta^2} \\ 0 & -\frac{\beta_x\beta_z}{\beta^2} & 1 - \frac{\beta_y\beta_z}{\beta^2} & -\frac{\beta_z^2}{\beta^2} \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1 + \alpha\beta_x^2 & \alpha\beta_y\beta_x & \alpha\beta_z\beta_x \\ -\gamma\beta_y & \alpha\beta_x\beta_y & 1 + \alpha\beta_y^2 & \alpha\beta_z\beta_y \\ -\gamma\beta_z & \alpha\beta_x\beta_z & \alpha\beta_y\beta_z & 1 + \alpha\beta_z^2 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Example 1.5

Writing down the transformation law from ZMF to S' which is the rest frame of the backward-moving particle

$$ct' = \gamma(ct - \beta x)$$
$$x' = \gamma(x - \beta ct)$$

Plug in $x = vt$

$$ct' = \gamma(c - \beta v)t = \frac{c^2 + v^2}{c^2\sqrt{1 - \frac{v}{c}^2}}ct$$
$$x' = \gamma(v - \beta c)t = \frac{2v}{\sqrt{1 - \frac{v}{c}^2}}t$$
$$\implies v' = \frac{x'}{t'} = \frac{2v}{1 + \frac{v}{c}^2}$$

Example 1.6

(a)

The direction of the rdv parallel to the direction of motion is contracted:

$$l_x = \gamma^{-1} l_0 \cos\theta'$$

The direction perpendicular to the motion is unchanged. That gives

$$\theta = \tan^{-1}\left(\frac{\gamma \sin\theta'}{\cos\theta'}\right)$$

(b)

Write down the transform rules in standard configuration and plug in $x' = u't'\cos\theta, y = u't'\sin\theta$:

$$\begin{pmatrix} ct \\ x \\ y \end{pmatrix} = \begin{pmatrix} \gamma & +\gamma\beta & 0 \\ +\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ u't'\cos\theta' \\ u't'\sin\theta' \end{pmatrix} = \begin{pmatrix} \gamma(c + \beta u'\cos\theta') \\ \gamma(u'\cos\theta' + \beta c) \\ u'\sin\theta' \end{pmatrix} t'$$

The angle observed in S frame is $\theta = \tan^{-1}\left(\frac{u'\sin\theta'}{\gamma(u'\cos\theta' + v)}\right)$. If the bullet was a photon, $\theta = \tan^{-1}\left(\frac{\sqrt{c^2-v^2}\sin\theta'}{c\cos\theta' + v}\right)$

Example 1.7

In S' frame, the angular distribution of photons is

$$P'(\theta')d\theta' = \frac{\sin\theta'}{2}d\theta'$$
$$P(0 \le \theta' \le \theta'_0) = -\frac{\cos\theta'}{2}\bigg|_0^{\theta'_0} = \frac{1-\cos\theta'_0}{2}$$

If θ is the angle that the photon makes with respect to the motion of the π-mesons. As computed in question 6.(b), the transformation rule of θ is $\theta = \tan^{-1}\left(\frac{\sqrt{c^2-v^2}\sin\theta'}{c\cos\theta' + v}\right)$. Applying reverse transform, $\theta' = \tan^{-1}\left(\frac{\sqrt{c^2-v^2}\sin\theta}{c\cos\theta - v}\right)$.

Substitute in P,

$$P(\theta) = -\frac{1}{2}\frac{\mathrm{d}\cos\theta'(\theta)}{\mathrm{d}\theta}$$

$$P(\theta) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}\sqrt{\frac{1}{1 + \frac{(c^2-v^2)\sin^2\theta}{c(\cos\theta - v)^2}}} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}\sqrt{\frac{(c\cos\theta - v)^2}{c^2\cos^2\theta - 2vc\cos\theta + v^2 + (c^2-v^2)\sin^2\theta}}$$

$$P(\theta) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{c\cos\theta - v}{c - v\cos\theta}\right)$$
$$P(\theta) = -\frac{1}{2}\left(\frac{-c\sin\theta(c - v\cos\theta) - v\sin\theta(c\cos\theta - v)}{(c - v\cos\theta)^2}\right)$$
$$P(\theta) = \frac{1}{2}\frac{\sin\theta(c^2 - v^2)}{(c - v\cos\theta)^2}$$
$$P(\theta) = \frac{\sin\theta}{2\gamma^2(1 - \beta\cos\theta)^2}$$

Example 1.8

(a)

$$\begin{pmatrix} c\,\mathrm{d}t' \\ \mathrm{d}x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}\begin{pmatrix} c\,\mathrm{d}t \\ \mathrm{d}x \end{pmatrix}$$

Here, β and γ denote constant factors at a specific time

$$a'_x = \frac{\mathrm{d}}{\mathrm{d}t'}\frac{\mathrm{d}x'}{\mathrm{d}t'} = c\frac{\mathrm{d}}{\mathrm{d}t'}\frac{\gamma u - \gamma\beta c}{\gamma c - \gamma\beta u}$$
$$a'_x = c\frac{\mathrm{d}}{\mathrm{d}t'}\frac{\gamma u - \gamma\beta}{\gamma c - \gamma\beta u}$$
$$a'_x = \frac{c^2}{(\gamma c - \gamma\beta u)}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\gamma u - \gamma\beta}{\gamma c - \gamma\beta u}$$
$$a'_x = \frac{1}{(1 - \frac{u}{c}^2)^{\frac{3}{2}}}a_x$$

Now we have the acceleration transform rules between the instantaneous rest frames of the moving spaceship and an inertial frame

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}t} &= \frac{1}{\gamma^3}f(\tau) \\
\frac{\mathrm{d}u}{\mathrm{d}\tau} &= \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}u}{\mathrm{d}t} \\
&= \frac{c}{\gamma^4(c - \beta u)}f(\tau) \\
&= \frac{1}{\gamma^2}f(\tau) \\
\frac{1}{1 - \frac{u}{c}^2}\frac{\mathrm{d}u}{\mathrm{d}\tau} &= f(\tau) \\
\int_0^\tau \mathrm{d}\tau c\frac{\mathrm{d}\tanh^{-1}\frac{u}{c}}{\mathrm{d}\tau} &= \int_0^\tau \mathrm{d}\tau f(\tau)
\end{aligned}$$

$$\begin{aligned} c\tanh^{-1}\frac{u}{c} - c\tanh^{-1}\frac{u_0}{c} &= c\psi(\tau) \\ \frac{u(\tau)-u_0}{1-\frac{u(\tau)u_0}{c^2}} &= c\tanh\psi(\tau) \end{aligned}$$

For $u(\tau)$ to reach c, any finite proper acceleration has to be supplied for a infinite period of time.

(b)

$$\begin{aligned} \int_0^{\tau_a} \mathrm{d}t(\tau)\, u &= \Delta x \\ \int_0^{\tau_a} \mathrm{d}\tau\, c\cosh\frac{g\tau}{c}\tanh\frac{g\tau}{c} &= \Delta x \\ \int_0^{\tau_a} \mathrm{d}\tau\, c\sinh\frac{g\tau}{c} &= \Delta x \\ \frac{c^2}{g}\left(\cosh\frac{g\tau_a}{c} - \cosh 0\right) &= \Delta x \\ \cosh\frac{g\tau_a}{c} &= \frac{g\Delta x}{c^2} + 1 \\ \tau_a &= 3.02 \text{ years (taking } g = 9.8\, m\, s^{-2}) \end{aligned}$$

Example 1.9

Constant x'^1 hypersurface equation in Cartesian coords: $x^1+x^2 = \text{const.}$ i.e. a plane parallel to $x_3 - axis$;

Constant x'^2 hypersurface equation in Cartesian coords: $x^1 - x^2 = \text{const.}$ i.e. another plane parallel to x_3-axis;

Constant x'^3 hypersurface equation in Cartesian coords: $x^3 - \frac{1}{2}\left[(x^1)^2 - (x^2)^2\right] = \text{const.}$ i.e. a surface constituted of stacked hyperbolae.

$$g'_{ab} = \delta_{cd}\frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2x'^2 & 2x'^2 & 1 \end{pmatrix}^T_{ac} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2x'^2 & 2x'^2 & 1 \end{pmatrix}_{cb}$$

$$g'_{ab} = \begin{pmatrix} 2+4(x'^2)^2 & 4x'^2x'^1 & 2x'^2 \\ 4x'^2x'^1 & 2+4(x'^1)^2 & 2x'^1 \\ 2x'^2 & 2x'^1 & 1 \end{pmatrix}_{ab}$$

In general $g_{ab} \neq 0$ for $a \neq b$, so the coordinate system is not orthogonal.

$$\mathrm{d}V = \sqrt{g}\,\mathrm{d}x'^1\,\mathrm{d}x'^2\,\mathrm{d}x'^3$$

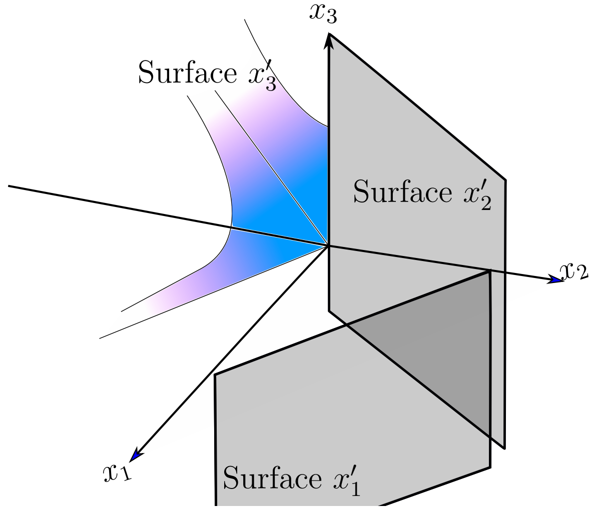


Figure 1: Sections of examples of such surfaces

$$\mathrm{d}V = \mathrm{d}x'^1\,\mathrm{d}x'^2\,\mathrm{d}x'^3\sqrt{2(2+4(x'^2)^2)-8(x'^2)^2}$$
$$\mathrm{d}V = 2\,\mathrm{d}x'^1\,\mathrm{d}x'^2\,\mathrm{d}x'^3$$

Example 1.10

$$x^2+y^2+z^2+w^2=a^2$$

$$w\,\mathrm{d}w = -\left(x\,\mathrm{d}x+y\,\mathrm{d}y+z\,\mathrm{d}z\right)$$
$$ds^2 = \mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2+\frac{(x\,\mathrm{d}x+y\,\mathrm{d}y+z\,\mathrm{d}z)^2}{a^2-x^2-y^2-z^2}$$

Let $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta, r = a\sin\chi$

$$ds^2 = \frac{a^2}{a^2=r^2}\,\mathrm{d}r^2+r^2\,\mathrm{d}\theta^2+r^2\sin^2\theta\,\mathrm{d}\phi^2$$
$$\mathrm{d}s^2 = a^2(\mathrm{d}\chi^2+\sin^2\chi(\mathrm{d}\theta\sin^2\theta\,\mathrm{d}\phi^2))$$

Metric for this 3D Riemannian space:

$$g_{ab} = a^2\begin{pmatrix}1 & & \\ & \sin^2\chi & \\ & & \sin^2\chi\sin^2\theta\end{pmatrix}$$

$$V = \iiint_{0,0,0}^{2\pi,\pi,\pi}\sqrt{a^6\sin^2\chi\sin^2\chi\sin^2\theta}\,\mathrm{d}\chi\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$
\begin{aligned}
&= a^3 2\pi \iint_{0,0}^{\pi,\pi} \sin^2\chi \sin\theta \,\mathrm{d}\chi \,\mathrm{d}\theta \\
&= 2\pi^2 a^3
\end{aligned}
$$

The embedded 2-Sphere defined by $\chi = \chi_0$ has line element

$$
\mathrm{d}s^2 = a^2 \sin^2\chi_0 (\mathrm{d}\theta^2 \sin^2\theta \,\mathrm{d}\phi^2)
$$

Therefore its metric is

$$
g_{ab} = a^2 \sin^2\chi_0 \begin{pmatrix} 1 & \\ & \sin^2\theta \end{pmatrix}
$$

The area is

$$
\begin{aligned}
A &= \iint_{0,0}^{2\pi,\pi} \sqrt{(a^2 \sin^2\chi_0)^2 \sin^2\theta} \,\mathrm{d}\theta \,\mathrm{d}\phi \\
&= 4\pi a^2 \sin^2\chi_0
\end{aligned}
$$

Example Sheet 2

Example 2.1

(a)

$$\begin{aligned}
\mathbf{e'_a} &= \frac{\partial}{\partial x'^a} \\
&= \frac{\partial x^b}{\partial x'^a}\mathbf{e_b} \\
\mathbf{e}'_1 &= \mathbf{e}_1 + \mathbf{e}_2 + 2x'^2\mathbf{e}_3 \\
\mathbf{e}'_2 &= \mathbf{e}_1 - \mathbf{e}_2 + 2x'^1\mathbf{e}_3 \\
\mathbf{e}'_3 &= \mathbf{e}_3
\end{aligned}$$

These are the tangent vectors to the intersections of the coordinate surfaces.

$$\mathbf{g}(\mathbf{e_a}, \mathbf{e_b}) = \delta_{ab}$$

$$\mathbf{g}(\mathbf{e'_a}, \mathbf{e'_b}) = \begin{pmatrix} 2 + 4(x'^2)^2 & 4x'^2x'^1 & 2x'^2 \\ 4x'^2x'^1 & 2 + 4(x'^1)^2 & 2x'^1 \\ 2x'^2 & 2x'^1 & 1 \end{pmatrix}_{ab} = g'_{ab}$$

Example 2.2

$\mathbf{v} = \mathbf{e_1}$

$$\mathbf{v} = v^a\mathbf{e}_a \qquad \Longrightarrow \qquad v^a = (1,0,0)^T, v_a = \delta_{ab}v^b = (1,0,0)$$

$$v'_a = \frac{\partial x^b}{\partial x'^a}v_b = (1,1,0)$$

$$v'^a = \frac{\partial x'^a}{\partial x^b}v^b = \left(\frac{1}{2}, \frac{1}{2}, -x'^1 - x'^2\right)$$

Example 2.3

(a)

$$\begin{aligned}
A^{ab}T_{ab} &= A^{ab}(T_{(ab)} + T_{[ab]}) \\
A^{ab}T_{ab} &= A^{ab}T_{(ab)} + A^{ab}T_{[ab]}
\end{aligned}$$

Using (anti)symmetry under exchange of dummy indices, we have

$$A^{ab}T_{(ab)} = -A^{ba}T_{(ba)} = 0$$

$$\implies \qquad A^{ab}T_{ab} = A^{ab}T_{[ab]}$$

Similarly,

$$S^{ab}T_{[ab]} = -S^{ba}T_{[ba]} = 0$$
$$S^{ab}T_{ab} = S^{ab}T_{(ab)}$$

(b)

$$\begin{aligned}
A'_{ab} &= \partial'_b v'_a - \partial'_a v'_b \\
&= \frac{\partial}{\partial x'^b}\left(\frac{\partial x^c}{\partial x'^a} v_c\right) - \frac{\partial}{\partial x'^a}\left(\frac{\partial x^c}{\partial x'^b} v_c\right) \\
&= \frac{\partial x^d}{\partial x'^b}\frac{\partial}{\partial x^d}\left(\frac{\partial x^c}{\partial x'^a} v_c\right) - \frac{\partial x^d}{\partial x'^a}\frac{\partial}{\partial x^d}\left(\frac{\partial x^c}{\partial x'^b} v_c\right) \\
&= \frac{\partial x^d}{\partial x'^b}\frac{\partial x^c}{\partial x'^a}\frac{\partial v_c}{\partial x^d} - \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}\frac{\partial v_d}{\partial x^c} + v_c\left(\frac{\partial x^d}{\partial x'^b}\frac{\partial^2 x^c}{\partial x^d \partial x'^a} - \frac{\partial x^d}{\partial x'^a}\frac{\partial^2 x^c}{\partial x^d \partial x'^b}\right) \\
&= \frac{\partial x^d}{\partial x'^b}\frac{\partial x^c}{\partial x'^a}\frac{\partial v_c}{\partial x^d} - \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}\frac{\partial v_d}{\partial x^c} + v_c\left(\frac{\partial^2 x^c}{\partial x'^b \partial x'^a} - \frac{\partial^2 x^c}{\partial x'^a \partial x'^b}\right) \\
&= \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b} A_{cd} \qquad \blacksquare
\end{aligned}$$

The components of A_{ab} does transform like a type-$(0,2)$ tensor.

$$B_{abc} = \frac{\partial A_{ab}}{\partial x^c} + \frac{\partial A_{bc}}{\partial x^a} + \frac{\partial A_{ca}}{\partial x^b}$$

$$\begin{aligned}
B'_{abc} &= \frac{\partial x^g}{\partial x'^c}\frac{\partial}{\partial x^g}\frac{\partial x^e}{\partial x'^a}\frac{\partial x^f}{\partial x'^b}A_{ef} + \frac{\partial x^g}{\partial x'^a}\frac{\partial}{\partial x^g}\frac{\partial x^e}{\partial x'^b}\frac{\partial x^f}{\partial x'^c}A_{ef} + \frac{\partial x^g}{\partial x'^b}\frac{\partial}{\partial x^g}\frac{\partial x^e}{\partial x'^c}\frac{\partial x^f}{\partial x'^a}A_{ef} \\
&= \frac{\partial x^g}{\partial x'^c}\frac{\partial x^e}{\partial x'^a}\frac{\partial x^f}{\partial x'^b}\frac{\partial A_{ef}}{\partial x^g} + \frac{\partial x^g}{\partial x'^a}\frac{\partial x^e}{\partial x'^b}\frac{\partial x^f}{\partial x'^c}\frac{\partial A_{ef}}{\partial x^g} + \frac{\partial x^g}{\partial x'^b}\frac{\partial x^e}{\partial x'^c}\frac{\partial x^f}{\partial x'^a}\frac{\partial A_{ef}}{\partial x^g} \\
&\quad + A_{ef}\left(\frac{\partial x^g}{\partial x'^c}\left(\frac{\partial^2 x^e}{\partial x^g \partial x'^a}\frac{\partial x^f}{\partial x'^b} + \frac{\partial^2 x'^f}{\partial x^g \partial x'^b}\frac{\partial x^e}{\partial x'^a}\right) + \frac{\partial x^g}{\partial x'^b}\cdots + \frac{\partial x^g}{\partial x'^a}\cdots\right)
\end{aligned}$$

$$\text{big chunky term} = A_{ef}\left(\left(\frac{\partial^2 x^e}{\partial x'^c \partial x'^a}\frac{\partial x^f}{\partial x'^b} + \frac{\partial^2 x'^f}{\partial x'^c \partial x'^b}\frac{\partial x^e}{\partial x'^a}\right) + \cdots + \dots\right)$$

but A_{ef} is antisymmetric in every frame by construction, so

$$\text{big chunky term} = A_{ef}\left(\left(\frac{\partial^2 x^e}{\partial x'^c \partial x'^a}\frac{\partial x^f}{\partial x'^b} - \frac{\partial^2 x'^e}{\partial x'^c \partial x'^b}\frac{\partial x^f}{\partial x'^a}\right) + \cdots + \dots\right)$$

denote $\dfrac{\partial^2 x^e}{\partial x'^c \partial x'^a}\dfrac{\partial x^f}{\partial x'^b}$ as Θ^{ef}_{cab}, Θ is symmetric under exchange of first two lower indices

$$\text{big chunky term} = A_{ef}\left(\Theta^{ef}_{cab} - \Theta^{ef}_{bca} + \Theta^{ef}_{abc} - \Theta^{ef}_{cab} + \Theta^{ef}_{bca} - \Theta^{ef}_{abc}\right) = 0$$

$$B'_{abc} = \frac{\partial x^g}{\partial x'^c}\frac{\partial x^e}{\partial x'^a}\frac{\partial x^f}{\partial x'^b}\frac{\partial A_{ef}}{\partial x^g} + \frac{\partial x^e}{\partial x'^a}\frac{\partial x^f}{\partial x'^b}\frac{\partial x^g}{\partial x'^c}\frac{\partial A_{fg}}{\partial x^e} + \frac{\partial x^f}{\partial x'^b}\frac{\partial x^g}{\partial x'^c}\frac{\partial x^e}{\partial x'^a}\frac{\partial A_{ge}}{\partial x^f}$$

$$B'_{abc} = \frac{\partial x^g}{\partial x'^c}\frac{\partial x^e}{\partial x'^a}\frac{\partial x^f}{\partial x'^b}\left(\frac{\partial A_{ef}}{\partial x^g} + \frac{\partial A_{fg}}{\partial x^e} + \frac{\partial A_{ge}}{\partial x^f}\right) = \frac{\partial x^g}{\partial x'^c}\frac{\partial x^e}{\partial x'^a}\frac{\partial x^f}{\partial x'^b}B_{abc}$$

B_{abc} is antisymmetric under exchange of any two indices.

Example 2.4

(a)

$$\begin{aligned}
g &= \det(g_{ab}) \\
\frac{1}{g}\partial_c g &= \mathrm{Tr}\left(g^{ab}\partial_c g_{bc}\right) \\
\partial_c g &= g g^{ab}\partial_c g_{ba} \\
\partial_c g &= g g^{ab}\partial_c g_{ab} \qquad \text{using symmetry of } g_{ab}
\end{aligned}$$

(b)

$$\begin{aligned}
\nabla_c g_{ab} &= \partial_c g_{ab} - \Gamma^d{}_{ca} g_{db} - \Gamma^d{}_{cb} g_{da} \\
&= \partial_c g_{ab} - \frac{1}{2} g^{de}\left[(\partial_c g_{ae} + \partial_a g_{ce} - \partial_e g_{ac}) g_{db} + (\partial_c g_{be} + \partial_b g_{ce} - \partial_e g_{bc}) g_{da}\right] \\
&= \partial_c g_{ab} - \frac{1}{2}\left[(\partial_c g_{ab} + \partial_a g_{cb} - \partial_b g_{ac}) + (\partial_c g_{ba} + \partial_b g_{ca} - \partial_a g_{bc})\right] \\
&= \partial_c g_{ab} - \frac{1}{2}[\partial_c g_{ab} - \partial_c g_{ba}] \\
&= 0
\end{aligned}$$

(c)

$$\Gamma^a_{bc} = \frac{1}{2} g^{ae}(\partial_b g_{ce} + \partial_c g_{be} - \partial_e g_{bc})$$

Turn off summation convention for the rest of this question

$$\Gamma^a_{bc} = \frac{1}{2}\sum_e g^{ae}(\partial_b g_{ce} + \partial_c g_{be} - \partial_e g_{bc})$$

Using g_{ab} is diagonal, we have

$$\Gamma^a_{bc} = \frac{1}{2} g^{aa} (\partial_b g_{cc} \delta_{ac} + \partial_c g_{bb} \delta_{ab} - \partial_a g_{bc} \delta_{bc})$$

For $a \neq b \neq c$,

$$\delta_{ab}, \delta_{bc}, \delta_{ac} = 0 \implies \Gamma^a_{bc} = 0$$

If two of the indices are the same, we can get

$$\Gamma^a_{ac} = \frac{1}{2} g^{aa} \partial_c g_{aa} = \Gamma^a_{ca} \qquad \Gamma^a_{bb} = -\frac{1}{2} g^{aa} \partial_a g_{bb}$$

If all three indices are the same,

$$\Gamma^a_{aa} = \frac{1}{2} g^{aa} \partial_a g_{aa}$$

But for diagonal matrices, the diagonal entry of the inverse metric is the reciprocal of the diagonal entry, i.e.

$$g^{aa} = g^{-1}_{aa}$$

We can thus rearrange into

$$\Gamma^a_{ac} = \partial_c \ln\left(\sqrt{|g_{aa}|}\right) = \Gamma^a_{ca} \qquad \Gamma^a_{bb} = -\frac{1}{2 g_{aa}} \partial_a g_{bb}$$

Example 2.5

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\phi^2$$

$$g_{ab} = \begin{pmatrix} 1 & \\ & \rho^2 \end{pmatrix}_{ab}$$

(a)

From the last question, we know that the only possible nonzero connection coefficients are

$$\begin{aligned} \Gamma^\rho_{\rho\phi} = \Gamma^\rho_{\rho\phi} &= \partial_\phi \ln(1) = 0 \\ \Gamma^\phi_{\rho\phi} = \Gamma^\phi_{\phi\rho} &= \partial_\rho \ln(\rho) = \frac{1}{\rho} \\ \Gamma^\rho_{\phi\phi} &= -\frac{1}{2} \partial_\rho \rho^2 = -\rho \\ \Gamma^\phi_{\rho\rho} &= -\frac{1}{2\rho^2} \partial_\phi 1 = 0 \end{aligned}$$

(b)

$$\begin{aligned}\nabla_a v^a &= \partial_a v^a + \Gamma^a_{ab} v^b \\ &= \partial_\rho v^\rho + \partial_\phi v^\phi + \frac{1}{\rho} v^\rho \\ &= \frac{v^\rho + \rho \partial_\rho v^\rho}{\rho} + \partial_\phi v^\phi \\ &= \frac{\partial_\rho(\rho v^\rho)}{\rho} + \partial_\phi v^\phi\end{aligned}$$

To translate this result in terms of an orthonormal basis vector, we use $\tilde{v}_\phi = \rho v_\phi$ such that $|v|^2 = v_\rho^2 + \tilde{v}_\phi^2$, and obtain[1]

$$\nabla'_a v'^a = \frac{\partial_\rho(\rho v^\rho)}{\rho} + \frac{1}{\rho}\partial_\phi \tilde{v}^\phi$$

(c)

Laplacian of a scalar field is given by

$$\begin{aligned}\nabla^2 f &= \nabla^a(\nabla_a f) \\ &= g^{ba}\nabla_b(\partial_a f) \\ &= \frac{\partial_\rho\big(\rho \partial_\rho f\big)}{\rho} + \frac{1}{\rho^2}\partial_\phi^2 f\end{aligned}$$

Example 2.6

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 g_{ab} = \begin{pmatrix} 1 & \\ & \sin^2\theta \end{pmatrix}_{ab}$$

(a)

Again we use the results from question 3. The only *possible* nonzero connection coefficients of this coordinate system are

$$\begin{aligned}\Gamma^\theta_{\phi\theta} &= \Gamma^\theta_{\theta\phi} = \partial_\phi \ln(1) = 0 \\ \Gamma^\phi_{\theta\phi} &= \Gamma^\phi_{\phi\theta} = \partial_\theta \ln(\sin\theta) = \cot\theta\end{aligned}$$

[1] $\tilde{v}_\phi$ is not a vector component, nor is the "normalised basis" a basis, in the sense that is usually used in this course.

$$\Gamma^{\phi}_{\theta\theta} = -\frac{1}{2\sin^2\theta}\partial_\phi 1 = 0$$
$$\Gamma^{\theta}_{\phi\phi} = -\frac{1}{2}\partial_\theta \sin^2\theta = -\sin\theta\cos\theta$$

(b)

$$\begin{aligned}
L &= g_{ab}\dot{x}^a\dot{x}^b \\
\frac{\partial L}{\partial x^c} &= \frac{\mathrm{d}}{\mathrm{d}u}\frac{\partial L}{\partial \dot{x}^c} \\
\frac{\partial g_{ab}}{\partial x^c}\dot{x}^a\dot{x}^b &= \frac{\mathrm{d}}{\mathrm{d}u} g_{ab}\left(\delta^a_c\dot{x}^b + \dot{x}^a\delta^b_c\right) \\
\frac{\partial g_{ab}}{\partial x^c}\dot{x}^a\dot{x}^b &= 2\frac{\mathrm{d}}{\mathrm{d}u}\left(g_{cb}\dot{x}^b\right)
\end{aligned}$$

On the surface of a sphere

$$\begin{aligned}
2\sin\theta\cos\theta\dot{\phi}^2 &= 2\frac{\mathrm{d}\theta}{\mathrm{d}u} \\
0 &= \ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 \\
0 &= 2\frac{\mathrm{d}}{\mathrm{d}u}\left(\sin^2\theta\dot{\phi}\right) \\
0 &= \sin^2\theta\left(\cot\theta\,\dot{\phi}\dot{\theta} + \ddot{\phi}\right)
\end{aligned}$$

As we would've obtained from (a).

$$\begin{aligned}
&\ddot{\theta} + \Gamma^{\theta}_{\phi\phi}\dot{\phi}^2 + 0 + 0 + \ldots = 0 \\
\text{and} \qquad &\ddot{\phi} + \Gamma^{\phi}_{\phi\theta}\dot{\phi}\dot{\theta} + 0 + 0 + \ldots = 0
\end{aligned}$$

For a circle of constant latitude on a sphere θ is a constant. For this to satisfy geodesic equations

$$\begin{aligned}
0 &= -\sin\theta\,\cos\theta\,\dot{\phi}^2 \\
0 &= \ddot{\phi}
\end{aligned}$$

which gives $\cos\theta = 0 \implies \theta = \frac{\pi}{2}$, the equator. In general u, the affine parameter is linear in ϕ.

($\sin\theta = 0$ is not accepted because the coordinate system is degenerate at the north and south poles.)

(c)

$$
\begin{aligned}
\mathbf{v} &= 1\mathbf{e}_\theta \\
\frac{\mathrm{D}v^a}{\mathrm{D}\phi} &= 0 \\
\frac{\mathrm{d}v^a}{\mathrm{d}\phi} + \frac{\mathrm{d}x^b}{\mathrm{d}\phi}\Gamma^a_{\ bc}v^c &= 0 \\
\frac{\mathrm{d}v^\theta}{\mathrm{d}\phi} + \frac{\mathrm{d}\theta}{\mathrm{d}\phi}\Gamma^\theta_{\ \theta c}v^c + \frac{\mathrm{d}\phi}{\mathrm{d}\phi}\Gamma^\theta_{\ \phi c}v^c &= 0 \\
\frac{\mathrm{d}v^\theta}{\mathrm{d}\phi} - \sin\theta\cos\theta v^\phi &= 0 \\
\frac{\mathrm{d}v^\phi}{\mathrm{d}\phi} + \Gamma^\phi_{\ \phi c}v^c &= 0 \\
\frac{\mathrm{d}v^\phi}{\mathrm{d}\phi} + \cot\theta\, v^\theta &= 0
\end{aligned}
$$

Solving the two equations and plug in initial conditions

$$
\begin{aligned}
\sin\theta\cos\theta v^\phi &= -\tan\theta\ddot{v}^\phi \\
-\cos^2\theta_0 v^\phi &= \ddot{v}^\phi \\
v^\phi &= A\sin(\phi\cos\theta_0) \\
v^\theta &= \sin\theta_0 A\big(1-\cos(\phi\cos\theta_0)\big) + 1 \\
\implies A &= -\frac{1}{\sin\theta_0} \\
v^\phi &= -\frac{1}{\sin\theta_0}\sin(\phi\cos\theta_0) \\
v^\theta &= \cos(\phi\cos\theta_0)
\end{aligned}
$$

After parallel transport, we will have

$$
v^\phi = -\frac{1}{\sin\theta_0}\sin(2\pi\cos\theta_0) \qquad\qquad v^\theta = \cos(2\pi\cos\theta_0)
$$

which is not the same as what we started with, but

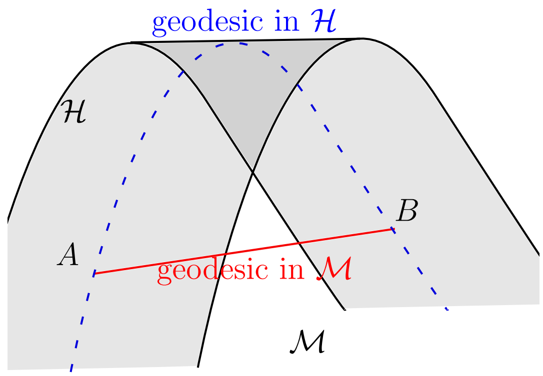
$$
v_a v^a = \left(v^\theta\right)^2 + \sin^2\theta_0\left(v^\phi\right)^2 = 1
$$

throughout the transport.

Example 2.7

If $\mathcal{C}$ is a geodesic in $\mathcal{M}$, the distance between the points along $\mathcal{C}$ is extremal among the set of distances of all other curves, that is, including the set of distances of other curves in $\mathcal{H}$. Therefore, $\mathcal{C}$ is also by definition a geodesic in $\mathcal{H}$.

The converse can be falsified by the following counterexample.



In Euclidean spacetime, the blue curve is a geodesic in $\mathcal{H}$ because it is the shortest path connecting A and B. However, it is not a geodesic in $\mathcal{M}$, as there are shorter paths connecting A and B.

Example 2.8

$$\text{hypersurface } \mathcal{H}\text{: } M \text{ dimensions}$$
$$\text{Euclidean space: } N > M \text{ dimensions}$$

(a)

Consider $\mathrm{d}s^2$ which is invariant,

$$\begin{aligned}
\mathrm{d}s^2 = \qquad \delta_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b &= g_{IJ}\,\mathrm{d}u^I\,\mathrm{d}u^J \\
\delta_{ab}\frac{\partial x^a}{\partial u^I}\frac{\partial x^b}{\partial u^J}\,\mathrm{d}u^I\,\mathrm{d}u^J &= g_{IJ}\,\mathrm{d}u^I\,\mathrm{d}u^J \\
g_{IJ} &= \delta_{ab}\frac{\partial x^a}{\partial u^I}\frac{\partial x^b}{\partial u^J}
\end{aligned}$$

(b)

Start with the explicit form of the metric connection

$$\Gamma^L_{JK} = \frac{\partial^2 x^a}{\partial u^J \partial u^k}\frac{\partial u^L}{\partial x^a}$$

$$g_{IL}\Gamma^L_{JK} = \delta_{bc}\frac{\partial x^b}{\partial u^I}\frac{\partial x^c}{\partial u^L}\frac{\partial^2 x^a}{\partial u^J \partial u^k}\frac{\partial u^L}{\partial x^a}$$
$$g_{IL}\Gamma^L_{JK} = \delta_{bc}\frac{\partial x^b}{\partial u^I}\frac{\partial^2 x^a}{\partial u^J \partial u^k}\delta^c_a$$
$$g_{IL}\Gamma^L_{JK} = \delta_{ab}\frac{\partial x^a}{\partial u^I}\frac{\partial^2 x^b}{\partial u^J \partial u^k}$$

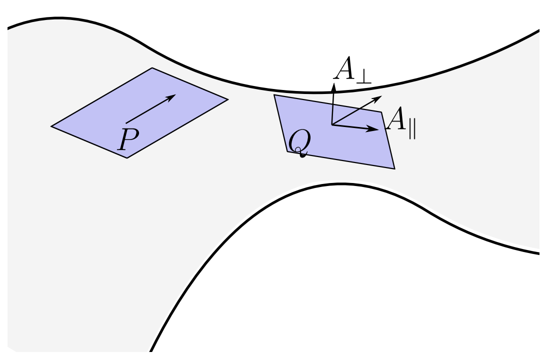
(c)

The vector **A** is invariant under coordinate transform, i.e.

$$A^I \mathbf{e}_I = A^a \mathbf{e}_a$$
$$A^I \frac{\partial}{\partial u^I} = A^b \frac{\partial}{\partial x^b}$$
$$A^I \frac{\partial x^a}{\partial u^I} = A^b \frac{\partial x^a}{\partial x^b}$$
$$A^I \frac{\partial x^a}{\partial u^I} = A^b \delta^a_b$$
$$A^I \frac{\partial x^a}{\partial u^I} = A^a$$

(d)

Given that the components of A are fixed in the embedding Euclidean space, we have



$$A^a(Q) = A^a(P) = A^I(P)\left.\frac{\partial x^a}{\partial u^I}\right|_P$$

The vector $A^a(Q)\mathbf{e}_a$ is not a vector in the hypersurface $\mathcal{H}$, but can be decomposed into components parallel and perpendicular to the tangent space at Q,

$$A^a(Q)\mathbf{e}_a = A^a_\parallel \mathbf{e}_a + \mathbf{A}^a_\perp e_a$$

where $A^a_{\parallel}\mathbf{e}_a$, lying in the tangent space, can be expressed as $A^I_{\parallel}(Q)\,\frac{\partial x^a}{\partial u^I}\Big|_Q$. Now we have

$$A^I(P)\,\frac{\partial x^a}{\partial u^I}\bigg|_P\mathbf{e}_a = A^I_{\parallel}(Q)\,\frac{\partial x^a}{\partial u^I}\bigg|_Q\mathbf{e}_a + A^a_{\perp}\mathbf{e}_a$$

Given the basis vectors are mutually orthogonal we can write the above as a vector equation

$$A^I(P)\,\frac{\partial x^a}{\partial u^I}\bigg|_P = A^I_{\parallel}(Q)\,\frac{\partial x^a}{\partial u^I}\bigg|_Q + A^a_{\perp}$$

Approximating to first order,

$$\begin{aligned}
A^I_{\parallel}(Q) &= A^I(P) + \delta A^I \\
\frac{\partial x^a}{\partial u^I}\bigg|_Q &= \frac{\partial x^a}{\partial u^I}\bigg|_P + \frac{\partial^2 x^a}{\partial u^I \partial u^J}\bigg|_P\,\delta u^J + O(\delta {u^J}^2) \\
0 &= \delta A^I\,\frac{\partial x^a}{\partial u^I}\bigg|_P + A^I(P)\,\frac{\partial^2 x^a}{\partial u^I \partial u^J}\bigg|_P\,\delta u^J + A^a_{\perp} \\
0 &= \delta_{ab}\delta A^I\frac{\partial x^a}{\partial u^I}\frac{\partial x^b}{\partial u^K} + \delta_{ab}A^I\frac{\partial^2 x^a}{\partial u^I \partial u^J}\frac{\partial x^b}{\partial u^K}\delta u^J
\end{aligned}$$

$$\begin{aligned}
\delta A^I\frac{\partial x^b}{\partial u^I}\frac{\partial x^b}{\partial u^K} &= -A^I\frac{\partial^2 x^b}{\partial u^I \partial u^J}\frac{\partial x^b}{\partial u^K}\delta u^J \\
g_{IK}\delta A^I &= -\delta_{ab}\frac{\partial x^b}{\partial u^K}\frac{\partial^2 x^a}{\partial u^I \partial u^J}A^I\delta u^J \\
g_{IK}\delta A^I &= -g_{KL}\Gamma^L_{IJ}A^I\delta u^J && \text{using (a)} \\
g_{IK}\delta A^I &= -g_{KI}\Gamma^I_{LJ}A^L\delta u^J && \text{where we swapped dummies } L \text{ and } I \\
\delta A^K &= -\Gamma^K_{JL}A^L\delta u^J && \text{relabeled } I \to K
\end{aligned}$$

The same as what we would've obtained from the parallel transport equation,

$$\delta A^K + \Gamma^K_{JL}(P)A^L(P)\,\delta u^J = \frac{\mathrm{D}A^K}{\mathrm{D}t}\delta t = 0$$

where t is an affine paramter for the curve along which the vector is transported.

Example 2.9

(a)

$$u^\mu = \frac{\mathrm{d}t}{\mathrm{d}\tau_{\mathcal{E}}}(c, \vec{u}) \qquad v^\mu = \frac{\mathrm{d}t}{\mathrm{d}\tau_{\mathcal{R}}}(c, \vec{v})$$

Since $u_\mu v^\mu$ is an invariant object we can always move to the frame where $\vec{u} = 0, |\vec{v}| = V$, where $\frac{\mathrm{d}t}{\mathrm{d}\tau_\mathcal{E}} = 1$

$$\begin{aligned}
&u_\mu v^\mu \\
=&\eta_{\mu\nu} u^\nu v^\nu \\
=&\frac{\mathrm{d}t}{\mathrm{d}\tau_V}\left(c^2 - \mathbf{0}\cdot\mathbf{V}\right) \\
=&\gamma_V c^2
\end{aligned}$$

(b)

The photon 4-momentum has expression

$$p^\mu = \frac{E}{c^2}\frac{\mathrm{d}x^\mu_\gamma}{\mathrm{d}t}$$

$u^\mu p_\mu$ is an invariant object, so we can simply evaluate it in the rest frame of $\mathcal{E}$.

$$\begin{aligned}
u^\mu p_\mu &= (c, 0)\left(\frac{E_\gamma}{c}, \vec{p}\right) \\
&= E_\gamma = h\nu_\mathcal{E}
\end{aligned}$$

Similarly

$$v^\nu p_\nu = h\nu_\mathcal{R}$$

and we have

$$\frac{\nu_\mathcal{E}}{\nu_\mathcal{R}} = \frac{u^\mu p_\mu}{v^\nu p_\nu}$$

Example 2.10

Proper acceleration is given by

$$\begin{aligned}
a^\mu &= \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} \\
&= \gamma_u \frac{\mathrm{d}}{\mathrm{d}t}\left[\gamma_u(c, \vec{u})\right] \\
&= \gamma_u\left[\gamma_u^3\frac{\mathbf{u}\cdot\mathbf{a}}{c^2}(c, \vec{u}) + \gamma_u(0, \mathbf{a})\right] \\
&= \gamma_u^4\frac{\mathbf{u}\cdot\mathbf{a}}{c^2}(c, \vec{u}) + \gamma_u^2(0, \vec{a}) \\
-\alpha^2 = a_\mu a^\mu &= \gamma_u^8\frac{(\mathbf{u}\cdot\mathbf{a})^2}{c^4}(c^2 - u^2) - 2\gamma_u^6\frac{(\mathbf{u}\cdot\mathbf{a})^2}{c^2} - \gamma_u^4\mathbf{a}\cdot\mathbf{a} \\
\alpha^2 &= \gamma_u^6\frac{(\mathbf{u}\cdot\mathbf{a})^2}{c^2} + \gamma_u^4\mathbf{a}\cdot\mathbf{a}
\end{aligned}$$

If the motion in S is circular with radius r, we will have

$$\mathbf{a} = \frac{u^2}{r}\hat{\mathbf{r}} \qquad\qquad \mathbf{u} \cdot \mathbf{a} = 0$$

which gives

$$\alpha = \frac{c^2 u^2}{(c^2 - u^2) r}$$

Example Sheet 3

Example 3.1

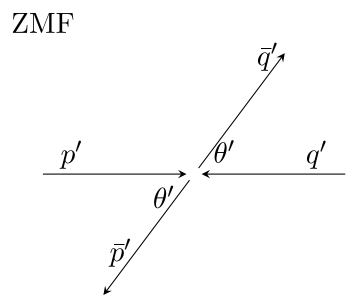
Let the four-momenta of the incident and stationary electrons before and after the collision in the lab frame be

$$p^\mu = (mc, \vec{0}) \qquad q^\mu = (\gamma_u mc, \vec{q}) \qquad \bar{p}^\mu = (\frac{\bar{E}_1}{c}, \vec{\bar{p}}) \qquad \bar{q}^\mu = (\frac{\bar{E}_2}{c}, \vec{\bar{q}})$$

respectively, we have conservation of 4 momenta throughout the process

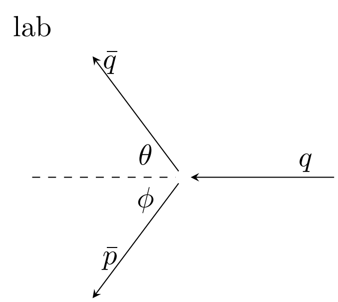
$$p^\mu + q^\mu = \bar{p}^\mu + \bar{q}^\mu$$

In the zero momentum S' frame, $\vec{p} = -\vec{q}$, and $\left|\vec{\bar{q}}\right| = |\vec{q}| = \left|\vec{\bar{p}}\right| = |\vec{p}|$, so we can draw



Write down the transform rules in with the incident particle velocity along x plug in $x'_q = u't'\cos\theta$, $y_q = u't'\sin\theta$:

$$\begin{pmatrix} ct \\ x_q \\ y_q \end{pmatrix} = \begin{pmatrix} \cosh\frac{\psi_u}{2} & +\sinh\frac{\psi_u}{2} & 0 \\ +\sinh\frac{\psi_u}{2} & \cosh\frac{\psi_u}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ u't'\cos\theta' \\ u't'\sin\theta' \end{pmatrix} = \begin{pmatrix} \cosh\frac{\psi_u}{2}(c + \beta u'\cos\theta') \\ \cosh\frac{\psi_u}{2}(u'\cos\theta' + \beta c) \\ u'\sin\theta' \end{pmatrix} t'$$



The angles observed in S frame have

$$\tan(\pi - \theta)\tan\phi = \frac{\sinh^2\left(\frac{\psi_u}{2}\right)\sin^2(\theta')}{\cosh^2\left(\frac{\psi_u}{2}\right)\sinh^2\left(\frac{\psi_u}{2}\right)\left(\cos^2(\theta') - 1\right)}$$

$$\tan\theta\tan\phi = \frac{1}{\cosh^2\left(\frac{\psi_u}{2}\right)}$$
$$\tan\theta\tan\phi = \frac{2}{\cosh^2\left(\frac{\psi_u}{2}\right) + \sinh^2\left(\frac{\psi_u}{2}\right) + 1}$$
$$\tan\theta\tan\phi = \frac{2}{\gamma_u + 1}$$

In the Newtonian limit for momentum and kinetic energy which is quadratic in momentum to be simulataneously conserved,

$$\bar{q}^2 + \bar{p}^2 = q^2 \qquad p_\perp\left(\frac{1}{\tan\theta} + \frac{1}{\tan\phi}\right) = q$$
$$\bar{p}^2\cos^2(\theta) + \frac{2p_\perp^2}{\tan\theta\tan\phi} + \bar{p}^2\cos^2(\phi) = \bar{p}^2 + \bar{q}^2$$
$$\frac{2p_\perp^2}{\tan\theta\tan\phi} = 2p_\perp^2$$
$$\tan\theta\tan\phi = 1$$

Which coincides with the limit $u \to 0$, $\frac{2}{\gamma_u+1} \to 1 - \frac{u^2}{4c^2} \approx 1$.

Example 3.2

In the mirror frame, the photon has 4-momentum (z-axis omitted)

$$p'^\mu = \begin{pmatrix} \frac{h\nu'}{c} \\ \frac{h\nu'}{c}\cos\theta' \\ \frac{h\nu'}{c}\sin\theta' \end{pmatrix}$$

which gives the invariant quantity

$$\eta_{\nu\mu}p^\nu_{\text{mirror}}p^\mu_{\text{photon}} = h\nu' m_{\text{mirror}} = h\nu\gamma_v m_{\text{mirror}}(1 - \beta\cos\theta)$$

where $\beta = \frac{v}{c}$, and the frequency shift

$$\nu' = \gamma_v\nu(1 + \beta\cos\theta)$$

After reflection, conserving energy and momentum parallel to the mirror plane,

$$\bar{p}'^\mu = \begin{pmatrix} \frac{h\nu'}{c} \\ -\frac{h\nu'}{c}\cos\theta' \\ \frac{h\nu'}{c}\sin\theta' \end{pmatrix}$$

A similar invariant quanity gives

$$\nu' = \gamma_v \bar{\nu}(1 - \beta\cos\phi)$$
$$\frac{\bar{\nu}}{\nu} = \frac{1+\beta\cos\theta}{1-\beta\cos\phi}$$

where ϕ is the reflected angle. Requiring the momentum component parallel to the mirror conserved in lab frame, we have

$$\bar{p}^{\nu} = \frac{h\bar{\nu}}{c}\begin{pmatrix} 1 \\ -\cos\phi \\ \sin\phi \end{pmatrix}$$
$$\frac{h\bar{\nu}}{c}\sin\phi = \frac{h\nu}{c}\sin\theta$$
$$\frac{\bar{\nu}}{\nu} = \frac{\sin\theta}{\sin\phi}$$
$$\frac{1-\beta\cos\phi}{\sin\phi} = \frac{1+\beta\cos\theta}{\sin\theta}$$
$$\sin\phi = \sin\theta\frac{1+\beta\cos\theta \pm \beta(\beta+\cos\theta)}{\beta^2+2\beta\cos\theta+1}$$
$$\sin\phi = \frac{\sin\theta}{\gamma_v(1+\beta^2+2\beta\cos\theta)}$$

So the reflected frequency is

$$\bar{\nu} = \gamma_v\big(1+\beta^2+2\beta\cos\theta\big)\nu$$

Example 3.3

Assume that a electron *did* emit a single photon. In the electron's initial rest frame

$$E_{\text{init}} = m_e c^2 \qquad p_{\text{init}} = 0$$
$$E_{\text{final}} = \sqrt{m_e c^2 + p_e^2 c^2} + h\nu \qquad p_{\text{final}} = \frac{h\nu}{c} - p_e$$

For both quantities to be conserved, the only solution for ν is 0, so no single photon can be emitted from an electron.

Similarly, assume that a massive *did* emit a single photon. In the particle's initial rest frame

$$E_{\text{init}} = mc^2 \qquad p_{\text{init}} = 0$$
$$E_{\text{final}} = h\nu \qquad p_{\text{final}} = \frac{h\nu}{c}$$

The two conservation conditions cannot be simultaneously satisfied, so no massive particle can decay into a single photon.

Example 3.4

(a)

The total 4-momentum is conserved, so

lab

$$p_1 = \gamma m_p u \longrightarrow \qquad \longleftarrow p_2 = -\gamma m_p u$$

$$\sum \vec{p}_{\text{after}} = 0 \qquad 2\gamma_u mc^2 = E_{p_1} + E_{p_2} + E_\pi$$

The minimum total kinetic energy for the reaction to occur is when $E_{p_1} = E_{p_2} = m_p c^2$, $E_\pi = m_\pi c^2$

$$E_{k,\min} = 2(\gamma_u - 1)m_p c^2 = m_\pi c^2$$

(b)

If one of the protons is stationary, denote the speed of the the incident proton v, and transfer to zero momentum frame, which is reduced to the scenario in (a).

$$E'_k = 2(\gamma_u - 1)m_p c^2 = m_\pi c^2$$

transform back into lab frame by a Lorentz boost of u_r,

$$\begin{aligned} \text{(ZMF energies)}\quad E'_1 = E'_2 &= \frac{m_\pi c^2}{2} + m_p c^2 = \cosh(\psi_u) m_p c^2 \\ E_1 &= \cosh(\psi_u + \psi_{u_r}) m_p c^2 \\ E_2 &= \cosh(\psi_u - \psi_{u_r}) m_p c^2 \end{aligned}$$

For one of the particles to become stationary, simply require $u_r = u$, which gives minimum kinetic energy in lab frame

$$\begin{aligned} E_k &= \gamma_v m_p c^2 - m_p c^2 \\ E_k &= \Big(2\cosh^2(\psi_u) - 1\Big) m_p c^2 - m_p c^2 \\ E_k &= \left[2\left(\frac{m_\pi}{2m_p} + 1\right)^2 - 1\right] m_p c^2 - m_p c^2 \\ E_k &= \left[\frac{m_\pi^2}{2m_p^2} + \frac{2m_\pi}{m_p} + 1\right] m_p c^2 - m_p c^2 \\ E_k &= \left(\frac{m_\pi}{2m_p} + 2\right) m_\pi c^2 \end{aligned}$$

Example 3.5

(a)

The second field equation consists of even permutations of $\sigma\mu\nu$, a field equation of odd permutations can be generated using antisymmetry of $F_{\mu\nu}$.

$$\begin{aligned} & & \partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} &= 0 \\ \text{antisymmetry} & \implies & -\partial_\sigma F_{\nu\mu} - \partial_\mu F_{\sigma\nu} - \partial_\nu F_{\mu\sigma} &= 0 \\ \text{sum together} & \implies & \partial_{[\sigma} F_{\mu\nu]} &= 0 \end{aligned}$$

(b)

The second field equation, in the form in (a), allows us to write $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, then the first equation can be written as

$$\begin{aligned} \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) &= \mu_0 j^\nu \\ \partial_\mu \partial^\mu A^\nu &= \mu_0 j^\nu \end{aligned}$$

Where Lorentz gauge $\partial_\mu A^\mu = 0$ was used. Definitions of the electric and magentic fields through $A^\mu = \left(\frac{\phi}{c}, \vec{A}\right)$ are

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \boldsymbol{\nabla}\phi \qquad \vec{B} = \boldsymbol{\nabla} \times \vec{A}$$

We derive Maxwell's equations one by one

$$\begin{aligned} \boldsymbol{\nabla} \cdot \vec{E} &= -\frac{\partial \boldsymbol{\nabla} \cdot \vec{A}}{\partial t} - \nabla^2 \phi \\ \boldsymbol{\nabla} \cdot \vec{E} &= \frac{1}{c^2} \frac{\partial}{\partial t} \frac{\partial \phi}{\partial t} - \nabla^2 \phi \\ \boldsymbol{\nabla} \cdot \vec{E} &= \partial_\mu \partial^\mu A^0 c \\ & \boxed{\boldsymbol{\nabla} \cdot \vec{E} = c^2 \mu_0 \rho = \frac{\rho}{\varepsilon_0}} \end{aligned}$$

$$\begin{aligned} \boldsymbol{\nabla} \cdot \vec{B} &= \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \vec{A} \\ \boldsymbol{\nabla} \cdot \vec{B} &= \partial_i \epsilon_{ijk} \partial_j A^k \\ \boldsymbol{\nabla} \cdot \vec{B} &= \epsilon_{ijk} \partial_i \partial_j A^k \\ & \boxed{\boldsymbol{\nabla} \cdot \vec{B} = 0} \end{aligned}$$

$$\begin{aligned} \boldsymbol{\nabla} \times \vec{B} &= \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times A) \\ \boldsymbol{\nabla} \times \vec{B} &= \vec{e}_i \epsilon_{kij} \partial_j \epsilon_{kmn} \partial_m A^n \\ \boldsymbol{\nabla} \times \vec{B} &= \vec{e}_i \big(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\big) \partial_j \partial_m A^n \\ \boldsymbol{\nabla} \times \vec{B} &= \boldsymbol{\nabla}\Big(\boldsymbol{\nabla} \cdot \vec{A}\Big) - \nabla^2 \vec{A} \end{aligned}$$

$$\begin{aligned} \boldsymbol{\nabla} \times \vec{E} &= -\frac{\partial \boldsymbol{\nabla} \times \vec{A}}{\partial t} - \boldsymbol{\nabla} \times \boldsymbol{\nabla} \cdot \phi \\ \boldsymbol{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} - \vec{e}_i \epsilon_{ijk} \partial_j \partial_k \phi \\ & \boxed{\boldsymbol{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}} \end{aligned}$$

$$\nabla \times \vec{B} = \frac{1}{c}\frac{\partial \nabla\phi}{\partial t} + \partial_\mu \partial^\mu \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2}$$
$$\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2}$$
$$\boxed{\nabla \times \vec{B} = \mu_0 \vec{J} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}}$$

(c)

The electric and magnetic fields

$$\vec{E} = -cF^{0i}\vec{e}_i \qquad \vec{B} = -\frac{1}{2}\epsilon_{ijk}F^{jk}\vec{e}_i \implies F^{ij} = -\epsilon^{ijk}B^k$$

are not tensors, but $F^{\mu\nu}$ is a tensor, so the components in two frames are related by

$$F'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma}$$

where

$$\Lambda^\rho{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}_{\rho\nu}$$

Working in natural units $c = 1$ to simplify expressions

$$F'^{ij} = \begin{pmatrix} -\beta\gamma E^1 & -\gamma E^1 & -\gamma E^2 + \beta\gamma B^3 & -\gamma E^3 - \beta\gamma B^2 \\ \gamma E^1 & -\beta\gamma E^1 & \gamma\beta E^2 - \gamma B^3 & \gamma\beta E^3 + \gamma B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -\gamma^2(1-\beta^2)E^1 & -\gamma(E^2 + \beta B^3) & -\gamma(E^3 - \beta B^2) \\ \gamma^2(1-\beta^2)E^1 & 0 & \gamma(\beta E^2 - B^3) & \gamma(\beta E^3 + B^2) \\ \gamma(E^2 - \beta B^3) & \gamma(B^3 - \beta E^2) & 0 & -B^1 \\ \gamma(E^3 + \beta B^2) & -\gamma(B^2 + \beta E^3) & B^1 & 0 \end{pmatrix}$$

Sub in $\gamma^2(1-\beta^2) = 1$. Reading off values for $\vec{E}$ and $\vec{B}$, and putting back c,

$$\vec{E} = \begin{pmatrix} E^1 \\ \gamma(E^2 - vB^3) \\ \gamma(E^3 + vB^2) \end{pmatrix} \qquad \vec{B} = \begin{pmatrix} B^1 \\ \gamma(B^2 + \frac{v}{c^2}E^3) \\ \gamma(B^3 - \frac{v}{c^2}E^2) \end{pmatrix}$$

(d)

The squared moduli of the fields are

$$
\begin{aligned}
\left|\vec{E}\right|^2 &= c^2 F^{0i} F^{0i} \\
\left|\vec{B}\right|^2 &= \frac{1}{4}\epsilon_{ijk}\epsilon_{imn} F^{jk} F^{mn} \\
\left|\vec{B}\right|^2 &= \frac{1}{4}\left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) F^{jk} F^{mn} \\
\left|\vec{B}\right|^2 &= \frac{1}{4}\left(F^{mk} F^{mk} - F^{nk} F^{kn}\right) \\
\left|\vec{B}\right|^2 &= \frac{1}{2} F^{mk} F^{mk} \\
F^{\mu\nu} F_{\mu\nu} &= F^{00} F_{00} + F^{0i} F_{0i} + F^{i0} F_{i0} + F^{mk} F_{mk} \\
F^{\mu\nu} F_{\mu\nu} &= 0 - 2\frac{\left|\vec{E}\right|^2}{c^2} + 2\left|\vec{B}\right|^2 \\
c^2\left|\vec{B}\right|^2 - \left|E^2\right| &= \frac{c^2 F^{\mu\nu} F_{\mu\nu}}{2}
\end{aligned}
$$

The speed of light and the contraction of two tensors are both invariant. Therefore, $c^2\left|\vec{B}\right|^2 - \left|E^2\right|$ is an invariant quantity.

Example 3.6

The spacetime interval of an infinitesimal section of the worldline of the satellite is invariant

$$
\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu
$$

In the weak-field approximation, $g_{00} \approx \left(1 + \frac{2\Phi}{c^2}\right) = -g_{11}$

$$
\begin{aligned}
\mathrm{d}s^2 = \left(1 + \frac{2\Phi(r)}{c^2}\right)\left(c^2\,\mathrm{d}t_0^2 - \mathrm{d}x_0^2\right) = c^2\,\mathrm{d}{\tau_C}^2 \\
\frac{1}{\gamma_u}\left(1 + \frac{2\Phi(r)}{c^2}\right)^{\frac{1}{2}}\mathrm{d}t_0 = \mathrm{d}\tau_C
\end{aligned}
$$

Where τ_C is the proper time measured by clock on the satellite, and t_0 the time measured at a point $\Phi = 0$ in Earth's rest frame S_0. Similarly, the proper time measured by the clock at North Pole, which is at rest in S_0 frame, satisfies

$$
\mathrm{d}s^2 = \left(1 + \frac{2\Phi(R)}{c^2}\right)\left(c^2\,\mathrm{d}t_0^2\right) = c^2\,\mathrm{d}{\tau_{C0}}^2
$$

$$\left(1+\frac{2\Phi(R)}{c^2}\right)^{\frac{1}{2}} \mathrm{d}t_0 = \mathrm{d}\tau_{C0}$$

Finally, substituting in $u^2 = \frac{GMm}{r}$ from Newtonian dynamics,

$$\begin{aligned}
\frac{\Delta\tau_C}{\Delta\tau_{C0}} &\approx \frac{1}{\gamma_u}\left(1+\frac{2\Phi(r)}{c^2}\right)^{\frac{1}{2}}\left(1+\frac{2\Phi(R)}{c^2}\right)^{-\frac{1}{2}} \\
&\approx \left(1+\frac{\Phi(r)}{c^2}\right)^{\frac{1}{2}}\left(1+\frac{2\Phi(r)}{c^2}\right)^{\frac{1}{2}}\left(1+\frac{2\Phi(R)}{c^2}\right)^{-\frac{1}{2}} \\
&\approx 1+\frac{1}{2}\left[\frac{\Phi(r)}{c^2}+\frac{2\Phi(r)}{c^2}-\frac{2\Phi(R)}{c^2}\right] \\
&\approx 1+\frac{3GMm}{2rc^2}-\frac{GMm}{Rc^2}
\end{aligned}$$

Example 3.7

The two line elements imply metrics

$$g_{ab} = \begin{pmatrix} x^2 & \\ & y^2 \end{pmatrix} \qquad \text{and} \qquad g_{ab} = \begin{pmatrix} y & \\ & x \end{pmatrix}$$

respectively. Exploiting the diagonality of the metrics, the only nonzero entries of the connections are

$$\begin{aligned}
\Gamma^a_{bc} &= \frac{1}{2}g^{ae}(\partial_b g_{ce} + \partial_c g_{be} - \partial_e g_{bc}) \\
\text{first manifold} \qquad \Gamma^x_{xx} &= \frac{1}{x},\ \Gamma^y_{yy} = \frac{1}{y} \\
\text{second manifold} \qquad \Gamma^x_{yy} &= -\frac{1}{2x},\ \Gamma^y_{xx} = -\frac{1}{2y}
\end{aligned}$$

yielding curvature tensors

$$\begin{aligned}
R_{abc}{}^d &= -\partial_a\Gamma^d_{bc} + \partial_b\Gamma^d_{ac} + \Gamma^e_{ac}\Gamma^d_{be} - \Gamma^e_{bc}\Gamma^d_{ae} \\
\text{first manifold} \qquad R_{xxx}{}^x &= 0,\ R_{yyy}{}^y = 0 && \implies R_{abc}{}^d = 0 \\
\text{second manifold} \qquad R_{xyy}{}^x &= -\partial_x\Gamma^x_{yy} = \frac{1}{2x^2} && \implies R_{abc}{}^d \neq 0
\end{aligned}$$

Therefore the first manifold is flat and the second is intrinsically curved.

Example 3.8

(a)

$$R_{abcd} = g_{de}\Big(-\partial_a \Gamma^e_{bc} + \partial_b \Gamma^e_{ac} + \Gamma^f_{ac}\Gamma^e_{bf} - \Gamma^f_{bc}\Gamma^e_{af}\Big)$$

Using the symmetries

$$R_{abcd} = -R_{bacd} \qquad R_{abcd} = R_{cdab} \qquad R_{[abc]d} = 0$$

In 2D there are 16 components in total,

$$\text{12 components of the form } R_{11\cdot\cdot} = R_{22\cdot\cdot} = R_{\cdot\cdot 11} = R_{\cdot\cdot 22} = 0$$
$$\text{Remaining 4 components are related by } R_{1221} = -R_{2121} = -R_{1212} = R_{2112}$$

Therefore on the 2-sphere there is only one independent component, which we can choose to be R_{1212}

$$\begin{aligned}
R_{\theta\phi\theta\phi} &= \sin^2\theta(-\partial_\theta \cot\theta + \partial_\phi 0 + 0 - \cot\theta\cot\theta) \\
&= \sin^2\theta\left(\frac{\sec^2\theta}{\tan^2\theta} - \frac{1}{\tan^2\theta}\right) \\
&= \sin^2\theta
\end{aligned}$$

(b)

The equation of geodesic deviation can be lowered to

$$g_{ea}\frac{\mathrm{D}}{\mathrm{D}u}\frac{\mathrm{D}\xi^e}{\mathrm{D}u} = R_{dbca}\frac{\mathrm{d}x^b}{\mathrm{d}u}\frac{\mathrm{d}x^c}{\mathrm{d}u}\xi^d$$

Substituting in $\xi^a = (0,\delta)^T$, $x^b = (\pi u, 0)^T$, the ϕ components of the left and tight hand sides are

$$\begin{aligned}
&g_{\phi\phi}\pi\left(\partial_\theta \frac{\mathrm{D}\xi^\phi}{\mathrm{D}u} + \Gamma^\phi_{\theta\phi}\frac{\mathrm{D}\xi^\phi}{\mathrm{D}u}\right) && R_{dbc\phi}\frac{\mathrm{d}x^b}{\mathrm{d}u}\frac{\mathrm{d}x^c}{\mathrm{d}u}\xi^d \\
&= \sin^2\theta\pi\left(\partial_\theta\pi\Big(\partial_\theta\delta + \Gamma^\phi_{\theta\phi}\delta\Big) + \Gamma^\phi_{\theta\phi}\pi\Big(\partial_\theta\delta + \Gamma^\phi_{\theta\phi}\delta\Big)\right) && = R_{\phi\theta\theta\phi}\frac{\mathrm{d}\theta}{\mathrm{d}u}\frac{\mathrm{d}\theta}{\mathrm{d}u}\delta \\
&= \sin^2\theta\pi^2\delta(\partial_\theta\cot\theta + \cot\theta\cot\theta) && = -\sin^2(\theta)\pi^2\delta \\
&= \sin^2\theta\pi^2\delta\left(-\frac{\sec^2\theta}{\tan^2\theta} + \cot\theta\cot\theta\right) && = -\sin^2(\theta)\pi^2\delta \\
&= -\sin^2\theta\pi^2\delta && = -\sin^2(\theta)\pi^2\delta
\end{aligned}$$

The θ components are

$$
\begin{aligned}
g_{\theta\theta}\pi\left(\partial_\theta\frac{\mathrm{D}\xi^\theta}{\mathrm{D}u}+\Gamma^\theta_{\theta d}\frac{\mathrm{D}\xi^d}{\mathrm{D}u}\right) & \qquad R_{dbc\theta}\frac{\mathrm{d}x^b}{\mathrm{d}u}\frac{\mathrm{d}x^c}{\mathrm{d}u}\xi^d \\
=\pi\left(\partial_\theta\pi\left(\partial_\theta 0+\Gamma^\theta_{\theta d}\frac{\mathrm{D}\xi^d}{\mathrm{D}u}\right)+0\right) & \qquad =0 \\
=0 & \qquad =0
\end{aligned}
$$

Indeed both components satisfy the equation of geodesic deviation.

Example 3.9

(a)

In Newtonian gravity,

$$
\begin{aligned}
\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}&=-\frac{\partial\phi}{\partial x^i}\\
\frac{\mathrm{d}^2\bar{x}^i}{\mathrm{d}t^2}&=-\frac{\partial\phi}{\partial\bar{x}^i}\\
\frac{\mathrm{d}^2\zeta^i}{\mathrm{d}t^2}&=-\left(\frac{\partial\phi}{\partial\bar{x}^i}-\frac{\partial\phi}{\partial x^i}\right)\\
\frac{\mathrm{d}^2\zeta^i}{\mathrm{d}t^2}&\approx-\zeta^j\frac{\partial}{\partial x^j}\left(\frac{\partial\phi}{\partial x^i}\right)\\
\frac{\mathrm{d}^2\zeta^i}{\mathrm{d}t^2}&\approx-\frac{\partial^2\phi}{\partial x^i\partial x^j}\zeta^j
\end{aligned}
$$

(b)

Starting with the equation of geodesic deviation, using $\frac{\mathrm{D}(\hat{e}_\alpha)^\mu}{\mathrm{D}\tau}=0$ for parallel transported vectors

$$
\begin{aligned}
\frac{\mathrm{D}}{\mathrm{D}\tau}\frac{\mathrm{D}\xi^\mu}{\mathrm{D}\tau}&=R_{\nu\alpha\beta}{}^\mu\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\xi^\nu\\
\frac{\mathrm{D}}{\mathrm{D}\tau}\frac{\mathrm{D}(\xi^{\hat{\alpha}}(\hat{e}_\alpha)^\mu)}{\mathrm{D}\tau}&=R_{\nu\alpha\beta}{}^\mu\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\xi^{\hat{\rho}}(\hat{e}_\rho)^\nu\\
\frac{\mathrm{D}}{\mathrm{D}\tau}\left[\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\left(\partial_\beta(\xi^{\hat{a}}(\hat{e}_\alpha)^\mu)+\Gamma^\mu_{\beta\nu}\xi^{\hat{a}}(\hat{e}_\alpha)^\nu\right)\right]&=R_{\nu\alpha\beta}{}^\mu u^\alpha u^\beta\xi^{\hat{\rho}}(\hat{e}_\rho)^\nu
\end{aligned}
$$

$$\frac{\mathrm{D}}{\mathrm{D}\tau}\left[\xi^{\hat{a}}\overbrace{\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\left(\partial_\beta(\hat{e}_\alpha)^\mu+\Gamma^\mu_{\beta\nu}(\hat{e}_\alpha)^\nu\right)}^{\frac{\mathrm{D}(\hat{e}_\alpha)^\mu}{\mathrm{D}\tau}=0}+(\hat{e}_\alpha)^\mu\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\partial_\beta\xi^{\hat{a}}\right]=R_{\nu\alpha\beta}{}^\mu u^\alpha u^\beta\xi^{\hat{\rho}}(\hat{e}_\rho)^\nu$$

$$\xi^{\hat{\alpha}}\frac{\mathrm{D}(\hat{e}_\alpha)^\mu}{\mathrm{D}\tau}+(\hat{e}_\alpha)^\mu\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\mathrm{d}\xi^{\hat{\alpha}}}{\mathrm{d}\tau}=R_{\nu\alpha\beta}{}^\mu u^\alpha u^\beta\xi^{\hat{\rho}}(\hat{e}_\rho)^\nu$$

$$(\hat{e}_\alpha)^\mu\frac{\mathrm{d}^2\xi^{\hat{\alpha}}}{\mathrm{d}\tau^2}=c^2R_{\nu\alpha\beta}{}^\mu(\hat{e}_0)^\alpha(\hat{e}_0)^\beta\xi^{\hat{\rho}}(\hat{e}_\rho)^\nu$$

As promised by Fermi, the general intrinsic derivative can be reduced to a simple derivative in a local-inertial coordinate system in the vicinity of a time-like geodesic.

(c)

In the weak field, time-independent Newtonian limit, assume $(\hat{e}_\alpha)^\mu\approx\delta^\mu_\alpha$, $\tau\approx t+O(\left(\frac{u}{c}\right)^2)$, $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$, the equation of geodesic deviation becomes

$$\frac{\mathrm{d}^2\xi^\mu}{\mathrm{d}t^2}\approx c^2R_{\nu00}{}^\mu\xi^\nu$$

$$\frac{\mathrm{d}^2\xi^\mu}{\mathrm{d}t^2}\approx\eta^{\gamma\mu}\frac{c^2}{2}\left(\partial_\nu\partial_\gamma h_{00}+\frac{1}{c^2}\partial_t\partial_t h_{\gamma\nu}-\frac{1}{c}\partial_t\partial_\gamma h_{0\nu}-\frac{1}{c}\partial_t\partial_\nu h_{\gamma0}\right)\xi^\nu$$

$$\frac{\mathrm{d}^2\xi^i}{\mathrm{d}t^2}\approx\frac{c^2}{2}\left[\frac{1}{c}(\partial_i\partial_t h_{00})\xi^t-\left(\partial_i\partial_j h_{00}\right)\xi^j\right]$$

$$\frac{\mathrm{d}^2\xi^i}{\mathrm{d}t^2}\approx-\frac{\partial^2(c^2h_{00}/2)}{\partial x^i\partial x^j}\xi^j$$

which is of the same form as the expression in (a).

Example Sheet 4

Example 4.1 Killing's equation

Given the metric components g'_{ab} are invariant under infinitesimal coordinate transformation $x^a \to x^a + \xi^a$, remembering $\nabla_a g_{bc} = 0$, retaining up to first order only,

$$\begin{aligned}
g'_{cd}(x')\,\mathrm{d}x'^c\,\mathrm{d}x'^d &= g_{ab}(x)\,\mathrm{d}x^a\,\mathrm{d}x^b \\
g'_{cd}(x') &= \frac{\partial x^a}{\partial x'^c}\frac{\partial x^b}{\partial x'^d} g_{ab}(x) \\
g'_{cd}(x') &= \left(\delta^a_c - \frac{\partial \xi^a}{\partial x^c}\right)\left(\delta^b_d - \frac{\partial \xi^b}{\partial x^d}\right) g_{ab}(x) \\
\text{invariance} \implies \qquad g_{cd}(x) + \xi^e \partial_e g_{cd} &= \left(\delta^a_c\delta^b_d - \frac{\partial \xi^a}{\partial x^c}\delta^b_d - \frac{\partial \xi^b}{\partial x^d}\delta^a_c\right) g_{ab}(x) \\
-\xi^e \partial_e g_{cd} &= \frac{\partial \xi^a}{\partial x^c} g_{ad} + \frac{\partial \xi^b}{\partial x^d} g_{cb} \\
-\xi^e \partial_e g_{cd} &= (\nabla_c \xi^a - \Gamma^a_{ce}\xi^e) g_{ad} + (\nabla_d \xi^b - \Gamma^b_{de}\xi^e) g_{cb} \\
\text{Rename some indices} \qquad -\xi^e \partial_e g_{cd} &= \nabla_c \xi_d + \nabla_d \xi_c - (g_{ad}\Gamma^a_{ce} + g_{ac}\Gamma^a_{de})\xi^e \\
&\boxed{\nabla_c \xi_d + \nabla_d \xi_c = 0}
\end{aligned}$$

where from the second-to-last line to the last line the following was used:

$$\begin{aligned}
\partial_e g_{cd} - g_{ad}\Gamma^a_{ce} - g_{ac}\Gamma^a_{de} &= \nabla_e g_{cd} = 0 \\
(g_{ad}\Gamma^a_{ce} + g_{ac}\Gamma^a_{de})\xi^e &= \xi^e \partial_e g_{cd}
\end{aligned}$$

If the spacetime metric is independent of x^0, we have $(\mathbf{e}_0)^a = \delta^a_0 \implies (\mathbf{e}_0)_b = g_{ba}\delta^a_0 = g_{b_0}$

$$\begin{aligned}
\nabla_b(\mathbf{e}_0)_a + \nabla_a(\mathbf{e}_0)_b &= \nabla_b g_{a0} + \nabla_a g_{b0} \\
&= \partial_b g_{a0} + \partial_a g_{b0} - \Gamma^c_{ba} g_{c0} - \Gamma^c_{ab} g_{c0} \\
&= \partial_b g_{a0} + \partial_a g_{b0} - \delta^d_0(\partial_a g_{db} + \partial_b g_{da} - \partial_d g_{ab}) \\
&= \partial_b g_{a0} + \partial_a g_{b0} - \partial_a g_{0b} - \partial_b g_{0a} + \underbrace{\partial_0 g_{ab}}_{0} \\
&= 0
\end{aligned}$$

Indeed $\mathbf{e}_0$ satisfies Killing's equation.

If $\mathbf{t}$ is the tangent vector to a geodesic affinely-parameterised by τ,

$$\begin{aligned}
t^a &= \frac{\mathrm{d}x^a}{\mathrm{d}\tau} \\
\frac{\mathrm{D}(\xi_a t^a)}{\mathrm{D}\tau} &= t^b\big(\nabla_b(\xi_a t^a)\big)
\end{aligned}$$

$$
\begin{aligned}
\frac{\mathrm{D}(\xi_a t^a)}{\mathrm{D}\tau} &= t^b t^a \nabla_b \xi_a + t^b \xi_a \nabla_b t^a \\
\frac{\mathrm{D}(\xi_a t^a)}{\mathrm{D}\tau} &= \overbrace{t^b t^a}^{\text{symmetric}} \quad \overbrace{\nabla_b \xi_a}^{\text{antisymmetric}} \quad = 0
\end{aligned}
$$

The intrinsic derivative of $\xi_a t^a$ is thus 0. Since $\xi_a t^a$ is a scalar, the intrinsic derivative coincides with the directional derivative along the curve. Therefore, $\xi_a t^a$ is constant along the geodesic.

Example 4.2 Dilation on a satellite

(a)

The Euler-Lagrange equation for the Lagrangian $L = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$ is

$$
\frac{\partial}{\partial x^\mu}\left[c^2\left(1-\frac{2\mu}{r}\right)\dot{t}^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2\right] = \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial}{\partial \dot{x}^\mu}\left[c^2\left(1-\frac{2\mu}{r}\right)\dot{t}^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2\right]
$$

$$
\begin{pmatrix} 0 \\ \frac{2\mu}{r^2}c^2\dot{t}^2 + \left(1-\frac{2\mu}{r}\right)^{-2}\frac{2\mu}{r^2}\dot{r}^2 - 2r(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \\ -2r^2\sin\theta\cos\theta\dot{\phi}^2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2c^2\left(1-\frac{2\mu}{r}\right)\ddot{t} + 2c^2\frac{2\mu}{r^2}\dot{r}\dot{t} \\ -2\left(1-\frac{2\mu}{r}\right)^{-1}\ddot{r} + 2\left(1-\frac{2\mu}{r}\right)^{-2}\frac{2\mu}{r^2}\dot{r}^2 \\ -4r\dot{r}\dot{\theta} - 2r^2\ddot{\theta} \\ -4r\dot{r}\sin^2\theta\dot{\phi} - 4r^2\sin\theta\cos\theta\dot{\theta}\dot{\phi} - 2r^2\sin^2\theta\ddot{\phi} \end{pmatrix}
$$

Formally, the geodesic equation is

$$
\begin{aligned}
\dot{x}^\mu \nabla_\mu \dot{x}^\nu &= 0 \\
\ddot{x}^\nu + \Gamma^\nu_{\mu\gamma}\dot{x}^\mu\dot{x}^\gamma &= 0
\end{aligned}
$$

The only nonzero coefficients can then be read off from the vector equation

$$
\Gamma^t_{rt} = \frac{1}{\frac{r^2}{\mu} - 2r}
$$

$$
\Gamma^r_{tt} = \frac{\mu c^2}{r^2}\left(1-\frac{2\mu}{r}\right) \qquad \Gamma^r_{rr} = -\left(1-\frac{2\mu}{r}\right)^{-1}\frac{\mu}{r^2}
$$

$$\Gamma^r_{\theta\theta} = -\left(1 - \frac{2\mu}{r}\right) r \qquad \Gamma^r_{\phi\phi} = -\left(1 - \frac{2\mu}{r}\right) r \sin^2\theta$$
$$\Gamma^\theta_{r\theta} = \frac{1}{r} \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$$
$$\Gamma^\phi_{r\phi} = \frac{1}{r} \qquad \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta}$$

(b)

The spacetime interval of an infinitesimal section of the worldline of the satellite is equal in all frames. Wlog, put the free falling satellite in a geodesic of constant r and constant $\theta = \frac{\pi}{2}$. The proper time on the satellite and the Schwarzschild metric time are related by

$$\begin{aligned} \mathrm{d}s^2 &= g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \\ c^2\,\mathrm{d}\tau^2 &= c^2\left(1 - \frac{2\mu}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}\mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\phi^2 \\ c^2\,\mathrm{d}\tau^2 &= c^2\left(1 - \frac{2\mu}{r}\right)\mathrm{d}t^2 - r^2\,\mathrm{d}\phi^2 \end{aligned}$$

The geodesic equation is

$$\begin{aligned} -\dot{\phi}^2 r\sin^2\theta + \dot{t}^2\frac{\mu c^2}{r^2} &= 0 \\ \dot{\phi}^2 &= \dot{t}^2\frac{\mu c^2}{r^3} \end{aligned}$$

Combining both above

$$\begin{aligned} c^2 &= \left[c^2\left(1 - \frac{2\mu}{r}\right) - \frac{\mu}{r}c^2\right]\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 \\ \frac{\mathrm{d}t}{\mathrm{d}\tau} &= \left(1 - \frac{3\mu}{r}\right)^{-\frac{1}{2}} \end{aligned}$$

The clock at rest at the north pole has $\theta = 0, \phi = 0, r = R$, its spacetime interval is

$$c^2\,\mathrm{d}{\tau_0}^2 = c^2\left(1 - \frac{2\mu}{R}\right)\mathrm{d}t^2$$

Therefore, the proper times of both clocks are related by

$$\frac{\Delta\tau}{\Delta\tau_0} = \frac{\mathrm{d}\tau}{\mathrm{d}\tau_0} = \left(1 - \frac{3\mu}{r}\right)^{\frac{1}{2}}\left(1 - \frac{2\mu}{R}\right)^{-\frac{1}{2}}$$

This result coincides with the weak field limit when $\mu = \frac{GM}{c^2}$.

Example 4.3 Red shift

The frequency and energy of the photon as observed by Bob the falling emitter are related by

$$h\nu_e = E_e = g_{\mu\nu} v^\mu(r_e) p^\nu$$

where p is the 4-momentum of the photon and $\mathbf{v} = \frac{\mathrm{d}\mathbf{x}_B}{\mathrm{d}\tau}$ is the 4-velocity of Bob. For a massive body like Bob, $v^\mu v_\mu = c^2$.

$$\begin{aligned}
c^2 &= g_{\mu\nu}(r) v^\mu(r) v^\nu(r) \\
c^2 &= \left(1 - \frac{2\mu}{r}\right) v^t v^t - \left(1 - \frac{2\mu}{r}\right)^{-1} v^r v^r
\end{aligned}$$

The geodesic equation satisfied by v^μ can be used to solve for the components of v, which can only be nonzero in r and t for a radially infalling particle.

$$\begin{aligned}
0 &= \frac{\mathrm{d}v^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\alpha\beta} v^\alpha v^\beta \\
0 &= \frac{r^2}{\mu} \frac{\mathrm{d}v^r}{\mathrm{d}\tau} + \left(1 - \frac{2\mu}{r}\right) v^t v^t - \left(1 - \frac{2\mu}{r}\right)^{-1} v^r v^r \\
0 &= \frac{r^2}{\mu} v^r \frac{\mathrm{d}v^r}{\mathrm{d}r} + c^2 \\
-\mu c^2 \frac{1}{r^2}\,\mathrm{d}r &= v^r\,\mathrm{d}v^r \\
v^r &= -\sqrt{2\mu c^2 \left(\frac{1}{r} - \frac{1}{R}\right)} \\
v^t &= c\left(1 - \frac{2\mu}{r}\right)^{-1} \sqrt{1 - \frac{2\mu}{R}}
\end{aligned}$$

The energy of the photon observed by Bob can now be explicitly written as

$$\begin{aligned}
E_e &= c\sqrt{1 - \frac{2\mu}{R}}\, p^t + c\left(1 - \frac{2\mu}{r}\right)^{-1} \sqrt{2\mu \frac{R-r}{Rr}}\, p^r \\
p^r p^r \left(1 - \frac{2\mu}{r}\right)^{-1} &= p^t p^t \left(1 - \frac{2\mu}{r}\right) \\
p^t &= p^r \left(1 - \frac{2\mu}{r}\right)^{-1} \\
E_e &= \left[\sqrt{1 - \frac{2\mu}{R}} + \sqrt{2\mu \frac{R-r}{Rr}}\right] \left(1 - \frac{2\mu}{r}\right)^{-1} c p^r
\end{aligned}$$

The photon momentum is parallel transported along its trajectory, using the fact t and r are the only varying coordinates,

$$
\begin{aligned}
0 &= \frac{\mathrm{d}p^\mu}{\mathrm{d}\lambda} + \Gamma^\beta_{\alpha\beta} p^\alpha p^\beta \\
\frac{\mathrm{d}p^r}{\mathrm{d}\lambda} &= -\Gamma^r_{tt} p^t p^t - \Gamma^r_{rr} p^r p^r \\
\frac{\mathrm{d}p^r}{\mathrm{d}\lambda} &= -\frac{\mu}{r^2}\left[\left(1-\frac{2\mu}{r}\right) p^t p^t - \left(1-\frac{2\mu}{r}\right)^{-1} p^r p^r\right] \\
\frac{\mathrm{d}p^r}{\mathrm{d}\lambda} &= 0
\end{aligned}
$$

where λ is an affine parameter such that $p^\lambda = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$ and in the last line $p^\mu p_\mu = 0$ was used. Therefore, p^r is a constant along the null geodesic. The energy of the photon observed by Alice is thus related to that observed by Bob by

$$
\begin{aligned}
p^t(R) &= \sqrt{-\frac{g_{rr}}{g_{tt}}} p^r_R = \left(1-\frac{2\mu}{R}\right)^{-1} p^r(R) \\
E_R &= p^t(R) u^t_R \\
u^t_R &= c\left(1-\frac{2\mu}{R}\right)^{-\frac{1}{2}} \\
E_R &= \left(1-\frac{2\mu}{R}\right)^{-\frac{1}{2}} c p_r \\
\frac{E_E}{E_R} &= \left(\sqrt{1-\frac{2\mu}{R}} + \sqrt{2\mu\frac{R-r}{Rr}}\right)\left(1-\frac{2\mu}{r}\right)^{-1}\left(1-\frac{2\mu}{R}\right)^{\frac{1}{2}}
\end{aligned}
$$

Example 4.4 Impact parameter

Starting with $p_\mu p^\mu = 0$ for a photon in the equatorial plane

$$\left(1-\frac{2\mu}{r}\right) c^2 \dot{t}^2 - \left(1-\frac{2\mu}{r}\right)^{-1} \dot{r}^2 - r^2\dot{\phi}^2 = 0$$

Since $g_{\mu\nu}$ is independent of t and ϕ, the first integrals of two corresponding components of the geodesic equation are constants

$$\left(1-\frac{2\mu}{r}\right)\dot{t} = k \qquad r^2\dot{\phi} = h$$

A light ray "grazes" the surface of a massive sphere at r, so $\dot{r}(r) = \frac{\mathrm{d}r}{\mathrm{d}\phi}\dot{\phi} = 0$. Substituting in,

$$0 = k^2\left(1-\frac{2\mu}{r}\right)^{-1} c^2 - \frac{h^2}{r^2}$$

$$\frac{h}{ck} = r\left(1 - \frac{2\mu}{r}\right)^{-\frac{1}{2}}$$

Assume $\phi \to 0$ (and hence $\dot{\phi} \to 0$) as $r \to \infty$. The impact parameter b is defined as

$$b \equiv \lim_{r\to\infty} r \sin\phi$$

Noticing

$$\begin{aligned} \lim_{r\to\infty}\left[k^2\left(1-\frac{2\mu}{r}\right)^{-1}c^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2 - \frac{h^2}{r^2}\right] &= 0 \\ \lim_{r\to\infty}\left[k^2c^2 - \dot{r}^2\right] &= 0 \\ \lim_{r\to\infty}\dot{r} &= ck \end{aligned}$$

The impact parameter can be calculated as

$$\begin{aligned} b &= \lim_{r\to\infty}\frac{\dot{\phi}\cos\phi}{\dot{r}/r^2} \\ b &= \lim_{r\to\infty}\frac{r^2\dot{\phi}}{\dot{r}} \\ b &= \frac{h}{ck} = r\left(1-\frac{2\mu}{r}\right)^{-\frac{1}{2}} \end{aligned}$$

For the sum, $M_\odot = 2\times 10^{30}$ kg, $R_\odot = 7\times 10^{8}$ m,

$$\begin{aligned} b - r &\approx \frac{r}{2}\frac{2\mu}{r} = \mu \\ b - r &\approx \frac{GM_\odot}{c^2} \\ b - r &\approx 2.97\times 10^{3}\ \mathrm{m} \end{aligned}$$

The light rays coming tangentially from the edge of the Sun will cast a image of radius b at infinity (ignoring diffraction). The Sun will seem bigger by about 3 km in radius.

Example 4.5 Schwarzschild blackhole

Consider the Schwarzschild metric

$$\mathrm{d}s^2 = \left(1-\frac{2\mu}{r}\right)c^2\,\mathrm{d}t^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\mathrm{d}r^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2$$

In region 2 where $r < 2\mu$, the proper time change $\Delta\tau$ between entering region 2 and reaching the origin

$$
\begin{aligned}
c^2\,\mathrm{d}\tau^2 &= \underbrace{\left(1-\frac{2\mu}{r}\right)c^2\,\mathrm{d}t^2}_{<0} - \left(1-\frac{2\mu}{r}\right)^{-1}\mathrm{d}r^2 - \underbrace{r^2\dot{\theta}^2}_{\leq 0} - \underbrace{r^2\sin^2\theta\dot{\phi}^2}_{\leq 0} \\
c\,\mathrm{d}\tau &< \left(\frac{2\mu}{r}-1\right)^{-\frac{1}{2}}(-\,\mathrm{d}r) \\
\Delta\tau &< -\frac{2\mu}{c}\int_1^0 \sqrt{\frac{r/2\mu}{1-r/2\mu}}\,\mathrm{d}\left(\frac{r}{2\mu}\right) \\
\Delta\tau &< \frac{2\mu}{c}\int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\cos\theta}2\sin\theta\cos\theta\,\mathrm{d}\theta \\
\Delta\tau &< \frac{\pi\mu}{c}
\end{aligned}
$$

is always less than $\frac{\pi\mu}{c}$.

Example 4.6

In an empty universe $T_{\mu\nu} = 0$ with vanishing cosmological constant $\Lambda = 0$, Einstein field equation dictates $R_{\mu\nu} = 0$.

$$
R_{\mu\nu} = -\partial_\rho\Gamma^\rho_{\mu\nu} + \partial_\mu\Gamma^\rho_{\rho\nu} + \Gamma^\rho_{\sigma\mu}\Gamma^\sigma_{\rho\nu} - \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\sigma\rho}
$$

The E-L euqations for the Lagrangian $L = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ are the geodesic equations of the given metric

$$
\begin{pmatrix}
-2c^2t\left[\dot{\chi}^2 + \sinh^2\chi\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right)\right] \\
-2c^2t^2\sinh\chi\cosh\chi\dot{\theta}^2 \\
-2c^2t^2\sinh^2\chi\sin\theta\cos\theta\dot{\phi}^2 \\
0
\end{pmatrix} =
$$

$$
\begin{pmatrix}
2c^2\ddot{t} \\
-2c^2t^2\ddot{\chi} - 4c^2t\dot{t}\dot{\chi} \\
-2c^2t\sinh^2\chi\left(t\ddot{\theta} + 2t\coth\chi\dot{\chi}\dot{\theta} + 2\dot{t}\dot{\theta}\right) \\
-2c^2t\sinh^2\chi\sin^2\theta\left(t\ddot{\phi} + 2\dot{t}\dot{\phi} + 2t\coth\chi\dot{\chi}\dot{\phi} + 2t\cot\theta\dot{\theta}\dot{\phi}\right)
\end{pmatrix}
$$

$$
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} =
\begin{pmatrix}
\ddot{t} + t\dot{\chi}^2 + t\sinh^2\chi\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) \\
\ddot{\chi} + \frac{2}{t}\dot{t}\dot{\chi} - \sinh\chi\cosh\chi\dot{\theta}^2 \\
\ddot{\theta} + 2\coth\chi\dot{\chi}\dot{\theta} + \frac{2}{t}\dot{t}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 \\
\ddot{\phi} + \frac{2}{t}\dot{t}\dot{\phi} + 2\coth\chi\dot{\chi}\dot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi}
\end{pmatrix}
$$

The independent components of the Ricci tensor of a diagonal metric are

$$
\begin{aligned}
R_{tt} &= -\partial_\rho \Gamma^\rho_{tt} + \partial_t \Gamma^\rho_{\rho t} + \Gamma^\rho_{\sigma t}\Gamma^\sigma_{\rho t} - \Gamma^\rho_{tt}\Gamma^\sigma_{\sigma\rho} \\
R_{tt} &= 0 + \partial_t\left(\frac{1}{t} + \frac{1}{t} + \frac{1}{t}\right) + 3\left(\frac{1}{t}\right)^2 - 0 \\
R_{tt} &= 0 \\
R_{\chi\chi} &= -\partial_\rho \Gamma^\rho_{\chi\chi} + \partial_\chi \Gamma^\rho_{\rho\chi} + \Gamma^\rho_{\sigma\chi}\Gamma^\sigma_{\rho\chi} - \Gamma^\rho_{\chi\chi}\Gamma^\sigma_{\sigma\rho} \\
R_{\chi\chi} &= -1 + 2\partial_\chi \coth\chi + 2(\coth\chi)^2 + 2 - 3t\frac{1}{t} \\
R_{\chi\chi} &= 0 \\
R_{\theta\theta} &= -\sinh^2\chi + \cosh^2\chi + \sinh^2\chi + \partial_\theta \cot\theta + 2t\sinh^2\chi\frac{1}{t} - 2t\cosh\chi + \cot^2\theta \\
&\quad - 3t\sinh^2\chi\frac{1}{t} + \sinh\chi\cosh\chi(\coth\chi + \coth\chi) \\
R_{\theta\theta} &= +\cosh^2\chi - \frac{1}{\sin^2\theta} + 2\sinh^2\chi + \cot^2\theta - 3\sinh^2\chi \\
R_{\theta\theta} &= 0 \\
R_{\phi\phi} &= -\partial_\rho \Gamma^\rho_{\phi\phi} + \partial_\phi \Gamma^\rho_{\rho\phi} + \Gamma^\rho_{\sigma\phi}\Gamma^\sigma_{\rho\phi} - \Gamma^\rho_{\phi\phi}\Gamma^\sigma_{\sigma\rho} \\
R_{\phi\phi} &= 0
\end{aligned}
$$

Therfore this metric satisfies $R_{\mu\nu} = 0$. The spatial hypersurfaces are hyperboloids.

Consider the transformation

$$\rho = ct\sinh\chi \qquad t' = t\cosh\chi$$

which takes the metric to

$$
\begin{aligned}
c^2\,\mathrm{d}t'^2 - \mathrm{d}\rho^2 &= c^2\left(\cosh^2\chi - \sinh^2\chi\right)\mathrm{d}t^2 + c^2t^2\sinh^2\chi\,\mathrm{d}\chi^2 - c^2t^2\cosh^2\chi\,\mathrm{d}\chi^2 \\
c^2\,\mathrm{d}t'^2 - \mathrm{d}\rho^2 &= c^2\,\mathrm{d}t^2 - c^2t^2\,\mathrm{d}\chi^2 \\
\mathrm{d}s^2 &= c^2\,\mathrm{d}t'^2 - \mathrm{d}\rho^2 - \rho^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)
\end{aligned}
$$

which is indeed the flat Minkowski spacetime expressed in spatial polar coordinates.

Example 4.7

Both velocities v_1 and v_2 satisfy

$$
\begin{aligned}
g_{\mu\nu}v^\mu v^\nu &= c^2 \\
\gamma_v^2c^2 - g_{ij}v^iv^j &= c^2 \\
g_{ij}v^iv^j &= (\gamma_v v)^2
\end{aligned}
$$

In an FRM universe, the metric takes the form

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\Big[\mathrm{d}\chi^2 + s^2(\chi)\Big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\Big)\Big]$$

. Wlog assume $v^\phi = v^\theta = 0$ The geodesic equation gives

$$\begin{aligned}\frac{\mathrm{d}v_\mu}{\mathrm{d}\tau} &= \frac{1}{2}(\partial_\mu g_{\alpha\beta})v^\alpha v^\beta \\ \frac{\mathrm{d}v_\chi}{\mathrm{d}\tau} &= 0\end{aligned}$$

The 4-velocity components are therefore related by

$$\begin{aligned}v^\chi &= g^{\chi\chi}v_\chi \\ \frac{\gamma_{v_1}v_1}{\gamma_{v_2}v_2} &= \frac{g_{\chi\chi}(t_2)}{g_{\chi\chi}(t_1)} \\ \frac{\gamma_{v_1}v_1}{\gamma_{v_2}v_2} &= \frac{a(t_2)}{a(t_1)}\end{aligned}$$

As $v_1 \to c$

$$\begin{aligned}\frac{h\nu_1}{h\nu_2}\frac{c}{c} &= \frac{p_1}{p_2} \\ \frac{\nu_1}{\nu_2} &= \frac{a(t_2)}{a(t_1)}\end{aligned}$$

Example 4.8

The Friedmann equations are

$$\begin{aligned}\frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{1}{3}\Lambda c^2 \\ \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G\rho}{3} + \frac{1}{3}\Lambda c^2 - \frac{Kc^2}{a^2}\end{aligned}$$

If $\Lambda = 0$, the conditions $\rho > 0$ and $p \geq 0$ mean that $\ddot{a} \neq 0$, i.e. the solution is nonstatic.

If $\Lambda \neq 0$, the first equation for a static, pressureless solution is

$$4\pi G\rho = \Lambda c^2$$

the second equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \Lambda c^2 - \frac{Kc^2}{a^2} = 0$$

$$a_{eq} = \sqrt{\frac{K}{\Lambda}}$$

Therefore a static solution exists. Conservation of energy and momentum implies $\rho a^3 = B$ where B is a constant. Substituting in,

$$\ddot{a} = -\frac{4\pi G}{3}\frac{B}{a^3} + \frac{1}{3}\Lambda c^2$$

At $a = a_{eq}$, $\frac{\mathrm{d}^2\ddot{a}}{\mathrm{d}a^2} < 0$, so the solution is unstable.